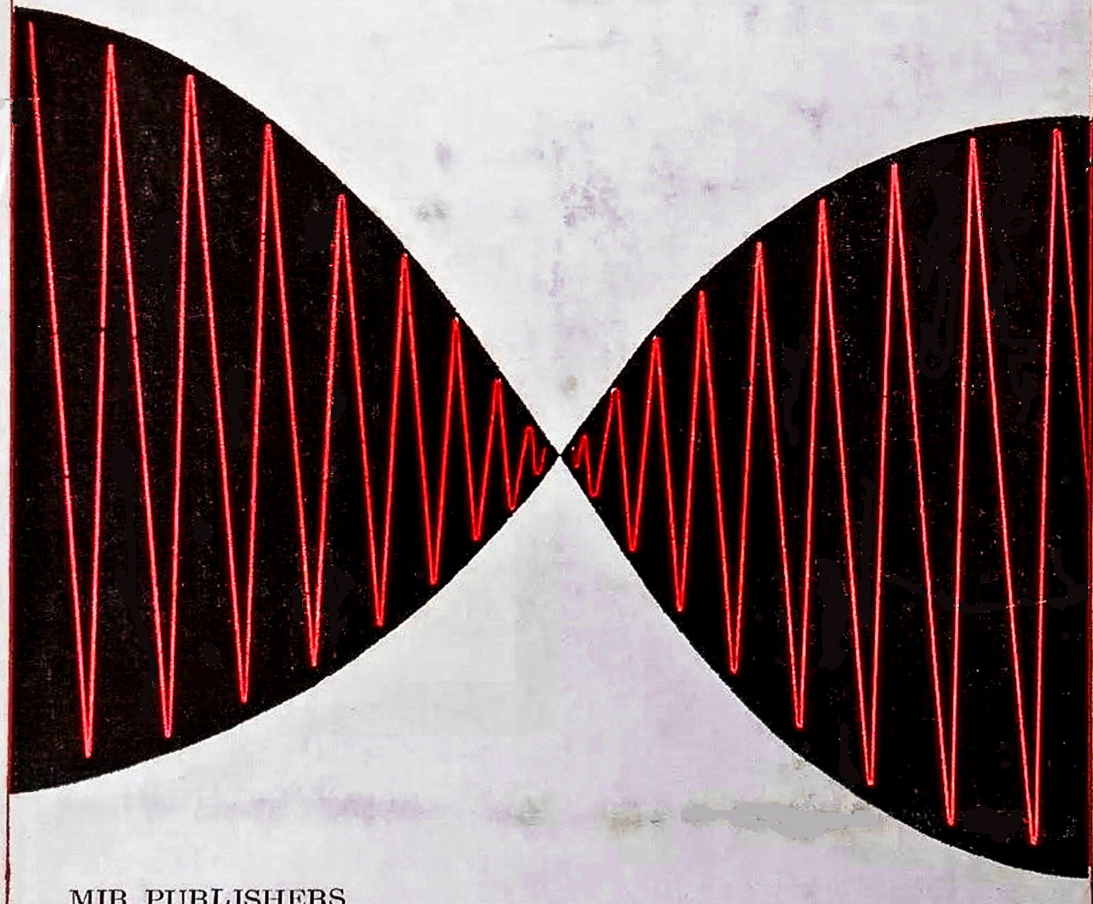
Fundamentals of Vibration Engineering

I. Bykhovsky



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И. И. БЫХОВСКИЙ

ОСНОВЫ ТЕОРИИ ВИБРАЦИОННОЙ ТЕХНИКИ

ИЗДАТЕЛЬСТВО «МАШИНОСТРОЕНИЕ»

FUNDAMENTALS OF VIBRATION ENGINEERING

Isidor I. Bykhovsky

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Preface

This is a book for engineers and scientific workers who are concerned with vibration engineering and technological applications of vibration, and for all those who have to study, teach, or solve problems in the dynamics of machines, instruments, and structures. In writing it the author has drawn on his many years' experience in the field and of lecturing and conducting seminars at teaching and research institutions and industrial establishments on the theory of vibrations and the dynamics of vibration machines.

Although many of the problems treated here are involved and subtle, they have been presented in a form that, we hope, makes them comprehensible to a wide readership. The information needed on the oscillatory motion of linear systems is briefly reviewed, and is followed by a description of the methods of investigating nonlinear systems and a treatment of linear and nonlinear problems in the dynamics of unbalanced-mass ('centrifugal') vibration generators, shock-and-vibration drives and vibrational processes, energy relations in vibrations, and the theory of dynamic vibration-control.

The author would like to thank all those who have actively assisted in the research and studies on which the book is based, or who contributed to their completion, and have thus, in a sense, participated in its writing. In particular he is grateful to L. B. Zaretsky, who accepted to write Sections 40, 41, 45, and 46, and parts of Sections 36 and 42. The author would also like to thank his colleagues A. D. Dorokhova, G. S. Klimovskaya, G. V. Kozlov, and I. S. Khaimchaev for their help with the preparation of the manuscript and drawings for the original Russian edition.

Certain additions have been made in this English edition to Section 36, while Sections 35 and 46 have been amended and some brief remarks and explanations have been introduced into several sections. The list of literature has been enlarged, and errata noticed in revising the Russian text have been corrected.

Moscow, January 1972 *I. I. Bykhovsky*

Introduction

Vibration engineering is concerned in the first place with machines, test beds and benches, devices, instruments, tools in which vibrations deliberately excited perform useful functions. In the exposition that follows this aspect will be our primary concern. Secondly, vibration engineering covers apparatus and arrangements for measuring and checking vibrations as well as for controlling them. Only in the third place can vibration engineering be regarded as concerned with devices for preventing, eliminating, damping, or isolating harmful vibrations.

Vibration engineering is based first and foremost on the theory of vibrations, especially the vibrations of nonlinear systems. This is mainly because many of the dynamic phenomena involved in the working of vibration generators, and nearly all vibrational processes, are essentially nonlinear. The theory of vibration engineering is also based on the theory of the stability of motion and in general on the successful application of qualitative methods of analysis of dynamic systems. It also makes use of the results obtained by the theory of automatic control, analytical mechanics, and other branches of science. It is closely related to the dynamics of machines and structures, acoustics, seismology, electrical engineering, ultrasonic techniques, and radio engineering, a relation determined primarily by the significance of oscillatory processes in all the fields mentioned.

Vibration machines have been devised, designed and perfected for various purposes by the assiduous efforts of many researchers, designers and engineers. At first problems were solved in a simplified way. The demands of technologists for higher efficiency were met by increasing the dimensions, weight, and power of equipment and the mass moment and angular velocity of the unbalance of centrifugal vibration generators.

Later, applying the simple ideas of the theory of the vibration of linear systems, attempts were made to design resonance or near-resonance machines, many of which proved unsuccessful, however, since the primitive theory did not take account of certain effects arising during their operation.

Large losses were incurred at the design stage by incorrect choice of power for the drives of machines with heavy vibratory loads, especially when the dissipative forces were essentially nonlinear.

The self-synchronization of two vibration generators operating on the same working member had been observed in many cases. This opened up encouraging prospects for considerably simplified machine designs with higher reliability. But attempts to 'tame' the, in essence nonlinear, phenomenon of self-synchronization empirically or by using linear schemes failed completely.

Adequate control of the dynamics of shock-and-vibration machines made imperative the application of the methods of analysing the stability of nonlinear systems with discontinuous conditions of motion. The design calculations of shock-and-vibration machines in fact made it necessary to apply nonlinear methods from the very beginning. Vibratory processes, as a rule, prove to be essentially nonlinear. Their analysis and the development of calculation techniques require the elaboration and application of subtle mathematical methods, work that is being successfully carried forward in the field of vibratory conveying.

A tendency may be observed to design new types of vibration machines whose action is based on the use of fine nonlinear dynamic effects or on methods of finding the optimum operating conditions and ensuring their stability. Their development is the result of prior theoretical work as, for example, in the case of machines with a superharmonic vibratory drive, of automation and self-tuning systems for shock-and-vibration and resonance vibration machines.

Many scientists and engineers have contributed to the solution of various individual theoretical problems. An outstanding contribution has been that of Prof. I. I. Blekhman. The theoretical basis of vibration engineering is still in the stage of growth and rapid development, but effective methods of analysis and synthesis of vibration equipment have already been evolved.

The two first chapters of this book are the minimum that must be grasped in order to read the rest. The other chapters and sections have been written, as far as possible, so as to make it feasible to use them separately (although, to avoid repetition, a number of cross references to other parts of the text have been introduced). In this manner the book has been given a certain reference character convenient for engineers who need to cope with separate problems quickly. For the same purpose a series of amplitude-frequency and phase-frequency curves have been included in Section 13. The author has attempted to eliminate the existing discrepancies in vibrational terminology and definitions. Certain theoretical problems have been omitted for lack of space and because of the need for detailed exposition in order to make certain other complicated but quite indispensable problems comprehensible.

CHAPTER 1

OSCILLATORY MOTION

1. Basic Concepts and Definitions

All the various processes of change of a scalar quantity may be grouped into two classes: oscillatory and non-oscillatory. The oscillatory process is characterized by alternate increases and decreases of a variable quantity. The non-oscillatory process has no such feature. It is convenient in some cases to consider the increases and decreases not absolutely but in relation to some other variable regarded as a reference level. The above is applicable to unidimensional motion where one coordinate alone is sufficient to specify the position of a moving object at a given time. The simplest case of such a motion is the movement of a point along a straight line. An oscillating point moves through all the positions (except for the extreme ones) in its path in opposite directions alternately.

Of special interest among oscillatory motions is the periodic motion, i.e., a motion which repeats itself in all its particulars at a definite equal interval of time. This time interval is termed the *period of vibration*. The function $f(t)$ expressing in mathematical form some process is called *periodic* if there exists a constant quantity T, the *period*, such that

$$f(t)=f(t\pm T)=f(t\pm 2T)=\ldots=f(t\pm nT) \tag{1}$$

where n is any positive integer.

If the motion of a point is two-dimensional (in a plane, for example) or three-dimensional (in space), other periodic motions are possible apart from the oscillatory one. For instance, a point can move periodically in one sense along a closed trajectory, say, a circle or an ellipse. This kind of motion may be called *circulating*[1]. In a circulating motion, unlike the oscillatory one, the moving point passes through all the positions on its path in one direction only. At the same time the oscillatory and circulating motions have much

---

[1] The term *circulating motion* is to be preferred to "cyclic motion", since the latter is often used to describe any periodic motion. For example, "one cycle of vibration" is a common expression. If the path is a circle, the term *rotational motion* is also used.

in common. The projections on the axes of rectilinear coordinates of a point moving along a circulating path perform oscillatory motions. This leads to a far-reaching analogy in general relationships and, hence, in the mathematical description of oscillatory and circulating motions. This has led to the use in physics and engineering of such terms as *conical pendulum*, *elliptical polarization* and *circular vibrations*.

Among many periodic vibrations the *sinusoidal* or *simple harmonic motion*[1] is of special interest; the oscillating quantity in this case is represented by a function sinusoidally varying in time, for example,

$$y = a \sin\left(\frac{2\pi}{T} t + \varphi\right) \tag{2}$$

where y = coordinate of a vibrating point measured from its middle position
a = amplitude of vibration displacement
T = period of vibration
φ = initial phase of vibration
t = running time value.

The *amplitude of vibration* is the absolute value of the maximum displacement of the vibrating point from its middle position in the case of a sinusoidal motion. The swing of vibration equal to twice the amplitude is the distance between two opposite extreme positions of the vibrating point. Strictly speaking, the term *amplitude* is not applicable to non-sinusoidal vibrations. In these cases one may speak of the peak values of an oscillating quantity or of the oscillation half-swing. For sinusoidal vibrations the terms *amplitude*, *peak value*, and *half-swing* are synonymous.

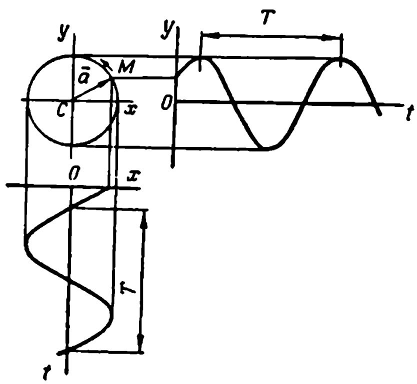

Figure 1

Sinusoidal vibrations can be conveniently represented by means of a rotating vector. If the radius-vector CM (Fig. 1) of constant length a is rotated uniformly, the projection of point M on any fixed diameter of the circle described by the point oscillates in a sinusoidal manner. If the angles are measured between the right-hand horizontal radius and the

[1] The term *sinusoidal motion* should be preferred since the *harmonic motion* is at present often used in a different sense, in distinction to subharmonic and superharmonic vibrations. The term *harmonic motion* is used for forced vibrations of any form provided the frequency is equal to that of the forcing factor.

radius-vector CM and φ is the initial value of the angle at $t = 0$ and the period of rotation of the vector CM is T, then the oscillations of the radius-vector projection on the vertical diameter are described by Eq. (2). The right-hand side of Fig. 1 shows the time-history, i.e., the oscillations versus time. The oscillations of the projection of the vector CM on the horizontal axis (Fig. 1) are described by

$$x = a\cos\left(\frac{2\pi}{T}t + \varphi\right) \tag{3}$$

The time-history is shown in the lower part of Fig. 1.

The angle $2\pi t/T + \varphi$ at which the vector CM is positioned at the moment t is called the *phase of vibrations*. The initial phase φ represents the phase at the origin of time, i.e., at $t = 0$. If a new origin of time is chosen, the initial phase φ changes but the phase $2\pi t/T + \varphi$ remains unchanged. Knowledge of the phase is needed to compare two or more oscillating quantities. In such cases the phase difference of the quantities being correlated may be of decisive importance. The phase is also of interest when the vibratory process is correlated with other phenomena such as a shock, connection to the system of supplementary links or constraints or their disconnection, etc. We then refer the phase to the moment at which the event occurs. Strictly speaking, the term *phase* is applicable only to sinusoidal oscillations.

The reciprocal of the period T is termed the *frequency* of vibration

$$f = \frac{1}{T} \tag{4}$$

The frequency f of vibration is generally measured in cycles per second (cps) or in Hertz (one cycle per second). In investigating vibrations it is often convenient to use the *angular frequency*

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{5}$$

which is the angular velocity of the radius-vector CM (Fig. 1)[1]. Using Eqs. (4) and (5), the phase of vibration can be expressed as follows:

$$\frac{2\pi}{T}t + \varphi = 2\pi f t + \varphi = \omega t + \varphi \tag{6}$$

The dimensions of the frequency f and the angular frequency ω are $\sec^{-1}$ or 1/sec. To avoid confusion, however, it is better to state the dimensions of frequency in Hz or cps (cycles per second) and that of angular frequency ω in rad/sec (radians per second). Often,

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[1] The terms *circular frequency* and *cyclic frequency* now falling into disuse seem to be less appropriate.

for the sake of brevity, when it is quite clear what is actually meant, the angular frequency is simply called frequency.

Sinusoidal vibrations of the same frequency are termed *synchronous vibrations*. For synchronous vibrations the phase difference is independent of the origin of time. If the phases of synchronous vibrations coincide[1], we say that two or more oscillating quantities are *in phase*. Two synchronous vibrations whose phases differ by π (by 180°) are said to be *in opposite phase* (or 180 out of phase, or antiphased).

2. Kinematics of Sinusoidal Vibration

Let the coordinate of an oscillating point be given by

$$x = x_a \cos(\omega t - \varphi) \tag{1}$$

where x_a = amplitude of displacement
$-\varphi$ = initial phase.

The quantity x is sometimes called a *vibration displacement*, particularly in vibration measurements.

Differentiating Eq. (1) with respect to time, we obtain the following expression for the velocity of the vibrating point (vibration velocity):

$$\frac{dx}{dt} \equiv \dot{x} = -\dot{x}_a \sin(\omega t - \varphi) = \dot{x}_a \cos\left(\omega t - \varphi + \frac{\pi}{2}\right) \tag{2}$$

where

$$\dot{x}_a = x_a \omega \tag{3}$$

is the amplitude of the vibration velocity. Comparing Eqs. (1) and (2), we find that for sinusoidal vibrations the velocity is ahead of the displacement by a phase angle of $\pi/2$ (90°).

Differentiating again with respect to time, we obtain an expression for the acceleration of the vibrating point (the acceleration of vibration):

$$\frac{d^2x}{dt^2} \equiv \ddot{x} = -\ddot{x}_a \cos(\omega t - \varphi) = \ddot{x}_a \cos(\omega t - \varphi + \pi) \tag{4}$$

where

$$\ddot{x}_a = \dot{x}_a \omega = x_a \omega^2 \tag{5}$$

is the vibration acceleration amplitude.

Note that the vibration acceleration is ahead of the velocity by a phase angle of $\pi/2$ and is in opposite phase to the displacement (see time-history, Fig. 2). It follows that displacement and acceleration

---

[1] Phases are said to coincide if their difference is any multiple of 2π.

reach their extreme values when the velocity is zero and are zero at extreme values of the velocity.

As noted in Section 1, the sinusoidal oscillation can be represented by using a uniformly rotating radius-vector of constant length. In Fig. 2 this vector is denoted by **r**. Hence the velocity is determined by the expression

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} = r\omega \mathbf{s}^0 \tag{6}$$

where $\mathbf{s}^0$ is the unit vector directed along the tangent line to the circle described by the end point of the radius-vector.

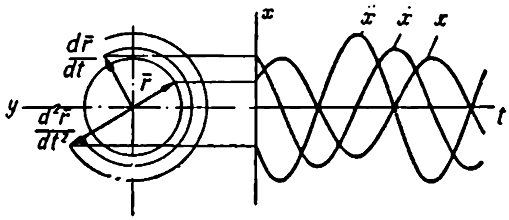

Figure 2

The velocity vector $d\mathbf{r}/dt$ is ahead of the radius-vector **r** by an angle of $\pi/2$. The acceleration is now expressed as follows:

$$\frac{d^2\mathbf{r}}{dt^2} = \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} = -r\omega^2 \mathbf{r}^0 \tag{7}$$

where $\mathbf{r}^0$ is the unit vector directed along the radius-vector **r**.

The acceleration vector $d^2\mathbf{r}/dt^2$ is ahead of the velocity vector $d\mathbf{r}/dt$ by an angle of $\pi/2$. The results obtained are similar to those given by Eqs. (2) and (4). The origins of vectors $d\mathbf{r}/dt$ and $d^2\mathbf{r}/dt^2$ in Fig. 2 have been brought to coincide with the origin of vector **r** to make easier their correlation with the time-history.

The radius-vector **r** of constant length rotating uniformly at the angular velocity ω can represent the oscillations of its projection $r \cos(\omega t - \varphi)$ onto the axis from which the angles are measured, or of the projection $r \sin(\omega t - \varphi)$ onto the axis at right angles to it. In general, unless otherwise specified, the projection meant is that on the axis from which the phase angles are measured[1].

It is advisable in some cases to use complex quantities to represent sinusoidal vibrations. Any complex quantity $z = x + iy$ can be represented in trigonometrical or exponential form:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{8}$$

where the modulus of the complex quantity

$$r = \sqrt{x^2 + y^2} \tag{9}$$

[1] In electrical engineering, usually the projection onto the perpendicular axis is considered.

and its argument measured from the positive direction of the x-axis

$$\theta = \tan^{-1}\frac{y}{x} \tag{10}$$

A sinusoidal vibration is represented by a complex quantity having a constant modulus r and an argument θ which is a linear function of time

$$\theta = \omega t - \varphi \tag{11}$$

The complex quantity

$$z = re^{i(\omega t - \varphi)} \tag{12}$$

may be used, like a rotating vector, to represent a sinusoidal vibration corresponding to its real part $\operatorname{Re} z = r\cos(\omega t - \varphi)$ or its imaginary part $\operatorname{Im} z = r\sin(\omega t - \varphi)$. Usually the real part is meant[1].

The vibration velocity is obtained by differentiating Eq. (12) with respect to time:

$$\frac{dz}{dt} \equiv \dot{z} = ir\omega e^{i(\omega t - \varphi)} = r\omega e^{i\left(\omega t - \varphi + \frac{\pi}{2}\right)} \tag{13}$$

After a second differentiation we obtain the vibration acceleration

$$\frac{d^2z}{dt^2} \equiv \ddot{z} = -r\omega^2 e^{i(\omega t - \varphi)} = r\omega^2 e^{i(\omega t - \varphi + \pi)} \tag{14}$$

Thus the results are the same as expressed by Eqs. (2) and (4) or (6) and (7). This was to be expected since the complex quantity z and the radius-vector $\mathbf{r}$ in a plane are in a one-to-one correspondence.

The complex quantity $z = re^{i(\omega t - \varphi)}$ and its conjugate $\bar{z} = re^{-i(\omega t - \varphi)}$ correspond to two vectors rotating at the same absolute value of angular velocity but in opposite senses, the former vector at the angular velocity ω and the latter at $-\omega$. The vibrations represented by the real parts of the two complex quantities are in phase and those represented by the imaginary parts are in opposite phase.

3. Addition of Vibrations

Consider the addition of collinear (i.e. performed along the same line) synchronous sinusoidal vibrations (Fig. 3a). Let there be n vibrations that are to be added:

$$x_i = a_i \cos(\omega t - \varphi_i), \qquad (i = 1, 2, \ldots, n) \tag{1}$$

According to Sec. 1, each vibration expressed by Eq. (1) can be represented by its vector. For convenience, the origin of the $(i+1)$st

[1] In electrical engineering, it is the imaginary part that is usually meant.

vector is made to coincide in Fig. 3*a* with the end of the *i*th vector. The sum of the vibrations $x = \sum_{i=1}^{n} x_i$ is also a sinusoidal vibration of the same frequency:

$$x = a\cos(\omega t - \varphi) \tag{2}$$

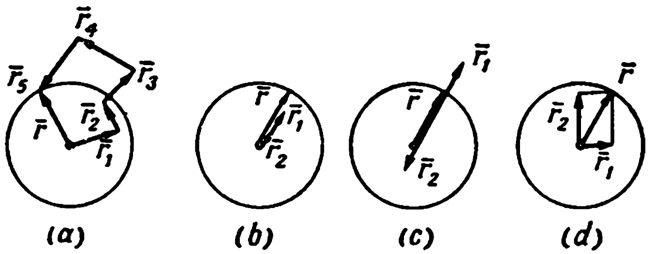

Figure 3

where, according to the rules of vector addition,

$$a = \sqrt{\left(\sum_{i=1}^{n} a_i \cos\varphi_i\right)^2 + \left(\sum_{i=1}^{n} a_i \sin\varphi_i\right)^2} \tag{3}$$

$$\varphi = \tan^{-1}\frac{\sum_{i=1}^{n} a_i \sin\varphi_i}{\sum_{i=1}^{n} a_i \cos\varphi_i} \tag{4}$$

If two collinear synchronous vibrations are to be added, then, on the basis of Eqs. (3) and (4), the amplitude and initial phase of the total motion can be written as follows:

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\varphi_2 - \varphi_1)} \tag{5}$$

$$\varphi = \varphi_1 + \tan^{-1}\frac{\sin(\varphi_2 - \varphi_1)}{\frac{a_1}{a_2} + \cos(\varphi_2 - \varphi_1)} \tag{6}$$

Three special cases resulting from formulas (5) and (6) should be noted:

1. The vibrations are in phase, i.e., $\varphi_2 = \varphi_1$ (Fig. 3*b*):

$$a = a_1 + a_2, \quad \varphi = \varphi_1 = \varphi_2 \tag{7}$$

2. The vibrations are in opposite phase, i.e., $\varphi_2 = \varphi_1 + \pi$ when $a_1 > a_2$ (Fig. 3*c*):

$$a = a_1 - a_2, \quad \varphi = \varphi_1 \tag{8}$$

3. The phase difference of the two vibrations is 90°, i.e., $\varphi_2 = \varphi_1 + \pi/2$ (Fig. 3*d*):

$$a = \sqrt{a_1^2 + a_2^2}, \quad \varphi = \varphi_1 + \tan^{-1}\frac{a_2}{a_1} \tag{9}$$

The addition of non-synchronous vibrations is a more complicated process. Consider the superposition of two collinear vibrations: $x_1 = a_1 \cos(\omega_1 t - \varphi_1)$ and $x_2 = a_2 \cos(\omega_2 t - \varphi_2)$ of different frequencies ω_1 and ω_2. The resulting motion $x = x_1 + x_2$ is not sinusoidal. However, considering the case when the frequencies of the two vibrations being superimposed are close to each other, i.e., $\omega_2 - \omega_1 \ll \omega_1$ (where $\omega_2 > \omega_1$), we can write their sum as $x = a \cos(\omega t - \varphi)$, where the "amplitude" a and the time-averaged "angular frequency" ω are variable quantities.

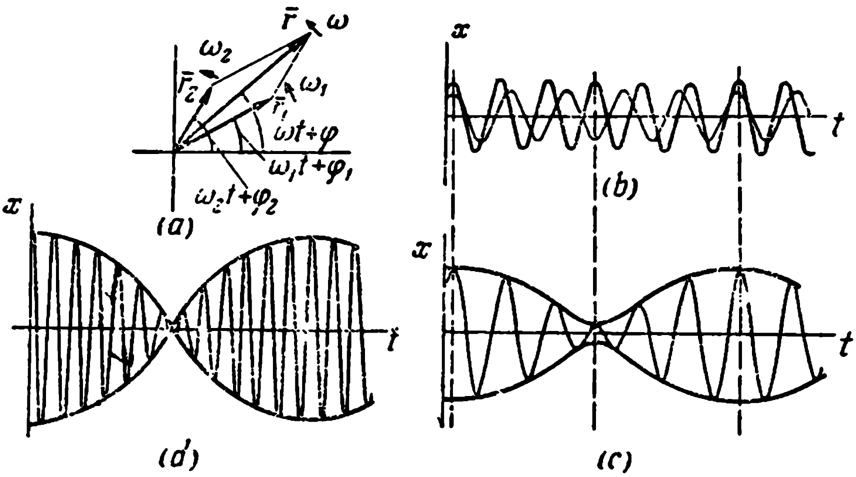

Figure 4

Using the vector representation (Fig. 4a), we obtain

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos[(\omega_2 - \omega_1)t - (\varphi_2 - \varphi_1)]} \tag{10}$$

$$\omega t = \omega_1 t +$$
$$+ \tan^{-1} \frac{a_1a_2\{\sin[(\omega_2 - \omega_1)t - (\varphi_2 - \varphi_1)] - \sin(\varphi_2 - \varphi_1)\} + a_2^2 \sin(\omega_2 - \omega_1)t}{a_1^2 + a_1a_2\{\cos[(\omega_2 - \omega_1)t - (\varphi_2 - \varphi_1)] + \cos(\varphi_2 - \varphi_1)\} + a_2^2 \cos(\omega_2 - \omega_1)t} \tag{11}$$

The running value of the "angular frequency" is

$$\frac{d(\omega t)}{dt} = \omega_1 + (\omega_2 - \omega_1) \cdot \frac{a_2^2 + a_1a_2 \cos[(\omega_2 - \omega_1)t - (\varphi_2 - \varphi_1)]}{a_1^2 + a_2^2 + 2a_1a_2 \cos[(\omega_2 - \omega_1)t - (\varphi_2 - \varphi_1)]} \tag{12}$$

The "initial phase" φ of the resulting motion is calculated from (6).

Figure 4b shows time-histories of vibrations x_1 and x_2. The time-histories of the resulting motion for the cases $a_1 \neq a_2$ and $a_1 = a_2$ are shown in Fig. 4c and d, respectively. In the latter case formula (10) is transformed into the following expression:

$$a = 2a_1 \left| \cos\left(\frac{\omega_2 - \omega_1}{2} t - \frac{\varphi_2 - \varphi_1}{2}\right) \right|$$

and, according to formula (12), the running value of the "angular frequency" proves constant and is equal to

$$\omega = \frac{\omega_1 + \omega_2}{2}$$

The phenomenon represented in Fig. 4*c* and *d* is known as *beats*.

The sum of two one-dimensional sinusoidal. coplanar (i.e., performed in the same plane) vibrations of equal frequency or frequencies forming a rational ratio is, in a general case, a periodic circulating motion. At some particular values of the difference of initial phases of the components the circulating motion degenerates into a periodic two-dimensional vibratory motion (or into a one-dimensional vibration in the case of two synchronous components).

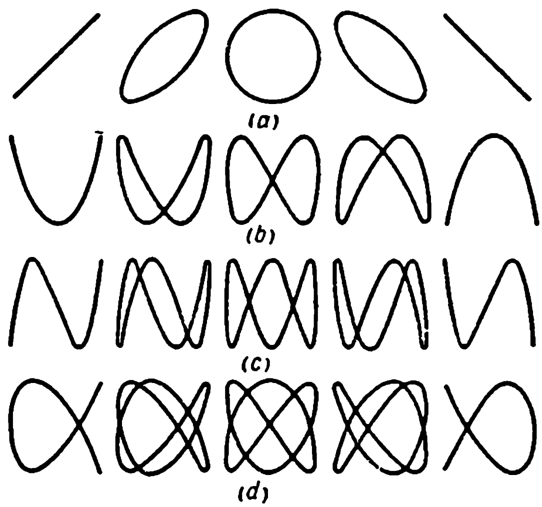

Figure 5

The path of a point performing simultaneously two such vibrations is termed a *Lissajou figure*. Figure 5*a* shows Lissajou figures corresponding to the addition of two synchronous vibrations at right angles to each other for various initial phase differences $\psi = \varphi_2 - \varphi_1$. The paths are the result of the addition of the vibrations $x = x_a \cos(\omega t - \varphi_1)$ and $y = y_a \cos(\omega t - \varphi_2)$. If two vibrations of unequal frequencies, $x = x_a \cos(n\omega t - \varphi_n)$ and $y = y_a \cos(m\omega t - \varphi_m)$, are being added up, then (as distinct from the case of synchronous vibrations) the difference of initial phases $\varphi_m - \varphi_n$ depends on the origin of time selected. In this case, it is convenient to use the reduced phase difference

$$\psi_{red} = \varphi_m - \frac{m}{n} \varphi_n \tag{13}$$

which is independent of the origin of time.

The Lissajou figures for various values of the reduced phase difference ($\psi_{red} = 0$; $\pi/4$; $\pi/2$; $3\pi/4$; π) with $m : n = 1 : 2$ are shown in Fig. 5*b*; with $m : n = 1 : 3$ in Fig. 5*c*; with $m : n = 2 : 3$ in Fig. 5*d*. The figures for $\psi_{red} = 5\pi/4$; $3\pi/2$; $7\pi/4$ coincide with those for $\psi_{red} = 3\pi/4$; $\pi/2$; $\pi/4$, respectively, but the direction of the traverse is reversed. If the ratio $m : n$ is irrational, the motion of the point

is then almost periodic (see Sec. 4) and its trajectory fills up completely a rectangle whose sides are $2x_a$, $2y_a$.

An interesting and practically important case is the addition of two circular circulating motions in phase, the points moving in opposite directions at equal radii r. One of these motions can be represented by $re^{i(\omega t-\varphi)}$ and the other by $re^{-i(\omega t-\varphi)}$. The resulting motion

$$re^{i(\omega t-\varphi)}+re^{-i(\omega t-\varphi)}=2r\cos(\omega t-\varphi) \tag{14}$$

is a one-dimensional vibration of the same frequency ω and amplitude $2r$.

4. Frequency Analysis of Vibration

The problem of vibration analysis is indeterminate: it permits many solutions. The uniqueness of a solution is obtained by imposing additional conditions. In what follows we shall discuss the analysis of collinear or scalar vibrations, our aim being to resolve the vibration into sinusoidal components.

Any bounded periodic function $f(t)$ having in its period $2\pi/\omega$ a finite number of extreme values and discontinuities can be expanded into a Fourier series:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t) \tag{1}$$

where

$$\left.\begin{aligned} a_n&=\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\cos n\omega t\,dt, \quad (n=0,1,2,\ldots)\\ b_n&=\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\sin n\omega t\,dt, \quad (n=1,2,\ldots)\end{aligned}\right\} \tag{2}$$

Expression (1) can be rewritten in the form:

$$f(t)=\frac{c_0}{2}+\sum_{n=1}^{\infty}c_n\cos(n\omega t-\varphi_n) \tag{3}$$

where

$$c_0=a_0;\ c_n=\sqrt{a_n^2+b_n^2};\ \varphi_n=\tan^{-1}\frac{b_n}{a_n},\quad (n=1,2,\ldots) \tag{4}$$

The terms of the series covered by the sign of summation in (3) are called *harmonics* or *tones*. The quantity c_n is the amplitude of the nth harmonic and $-\varphi_n$ is the initial phase of the same harmonic.

The operation of resolving a function into sinusoidal components is called *harmonic analysis*.

The limits of integration in formulas (2) may have another form provided the integration is carried out over a time interval equal to the period of the function $f(t)$. Thus, if the lower limit is ψ/ω, the upper limit must be $(2\pi+\psi)/\omega$. If $f(t)$ is an even function, i.e., $f(t) = f(-t)$, then all the coefficients b_n are zeros and the series (1) contains only cosine terms (it may also contain a constant component). If $f(t)$ is an odd function, i.e., $f(t) = -f(-t)$, then all coefficients a_n are zeros and only the sine terms remain in the series. A proper choice of the origin of time may, in some cases, render the function being analysed even or odd, which simplifies the calculations involved in the harmonic analysis.

The number of harmonics of some periodic functions may prove finite, but in general the Fourier series is infinite. At any point where the function $f(t)$ is continuous, its Fourier series converges to the value of $f(t)$. If at point $t = t_1$ the function $f(t)$ has a discontinuity, its Fourier series converges at this point to the mean of the function values on the right and left sides, i.e., to $\frac{1}{2}[f(t_1 - 0) + f(t_1 + 0)]$.

Note that for discontinuous functions the jump of the Fourier-series sum is higher than the jump of the function $f(t)$ by about 18% (Gibbs' phenomenon). If the discontinuous function $f(t)$ is approximated by a truncated Fourier series, i.e., by a finite number of its first terms, then the swing is found to increase in the neighbourhood of the discontinuity point and this is accompanied by high-frequency oscillations which decay as the distance from the point is increased. The set of the amplitudes c_n is called the *amplitude spectrum* of the function $f(t)$, and the set of the phases φ_n is known as its *phase spectrum*. The quantity $c_0/2$ is the mean value of the function $f(t)$. It is sometimes called the *constant component*.

Consider, as an example, an even periodic function which is described within one of the periods by the following relations:

$$f(t) = \begin{cases} t & \text{at } 0 \leqslant t \leqslant \frac{\pi}{\omega} \\ \frac{2\pi}{\omega} - t & \text{at } \frac{\pi}{\omega} \leqslant t \leqslant \frac{2\pi}{\omega} \end{cases}$$

This function is represented by the graph shown in Fig. 6*a*. From formulas (2) we obtain

$$a_0 = \frac{\pi}{\omega}, \quad a_n = \frac{2}{\pi n^2 \omega}[(-1)^n - 1], \quad b_n = 0$$

Hence

$$f(t) = \frac{\pi}{2\omega} - \frac{4}{\pi\omega}\left(\cos\omega t + \frac{1}{3^2}\cos 3\omega t + \frac{1}{5^2}\cos 5\omega t + \dots\right)$$

The amplitude spectrum of the function is shown in Fig. 6*b*.

As a second example, consider the harmonic analysis of an odd periodic function described within one of the periods by the relationship

$$f(t)=t \text{ at } -\frac{\pi}{\omega}\leqslant t\leqslant\frac{\pi}{\omega}$$

Figure 6*c* is a graphical representation of this function. Using formulas (2) and integrating between $-\pi/\omega$ and π/ω, we obtain

$$a_n=0,\ b_n=\frac{2}{n\omega}(-1)^{n-1}$$

whence

$$f(t)=\frac{2}{\omega}\left(\sin\omega t-\frac{1}{2}\sin 2\omega t+\frac{1}{3}\sin 3\omega t-\frac{1}{4}\sin 4\omega t+\dots\right)$$

The amplitude spectrum of this function is given in Fig. 6*d*.

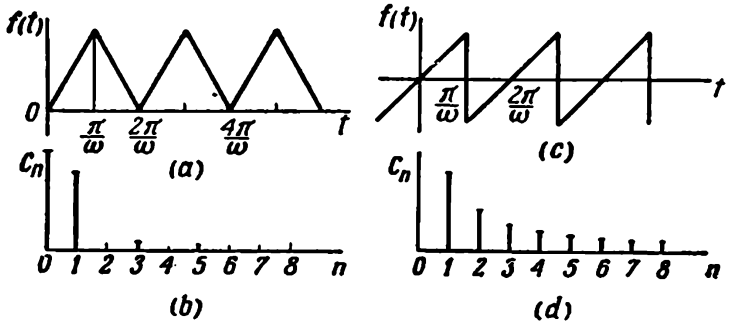

Figure 6

The less smooth is the function $f(t)$, the slower the Fourier series converges. Hence, the smoother the periodic function, the poorer it is in higher harmonics. As the latter of the functions considered above is discontinuous, its Fourier series converges more slowly than that of the former continuous function.

The spectrum of a periodic function is a line spectrum, i.e., it consists of separate lines corresponding to the amplitudes of the component harmonics. Non-periodic functions have a continuous spectrum. This means that the spectrum of a non-periodic function comprises, generally speaking, oscillations of all frequencies from 0 to ∞. The spectrum of a non-periodic function is determined by the Fourier integral rather than by the Fourier series.

The function $f(t)$ which is defined within the interval $-\infty < t < \infty$ and is continuous within the interval or has a finite number of discontinuities of the first kind on any finite segment and is absolutely integrable within the interval $-\infty < t < \infty$, i.e., satisfies the inequality

$$\int_{-\infty}^{\infty}|f(t)|\,dt<\infty$$

can be represented by the Fourier integral

$$f(t)=\frac{1}{\pi}\int_{0}^{\infty}d\omega\int_{-\infty}^{\infty}f(t_1)\cos\omega(t-t_1)\,dt_1=$$
$$=\frac{1}{\pi}\int_{0}^{\infty}\cos\omega t\,d\omega\int_{-\infty}^{\infty}f(t_1)\cos\omega t_1\,dt_1+$$
$$+\frac{1}{\pi}\int_{0}^{\infty}\sin\omega t\,d\omega\int_{-\infty}^{\infty}f(t_1)\sin\omega t_1\,dt_1=$$
$$=\frac{1}{\pi}\int_{0}^{\infty}[a(\omega)\cos\omega t+b(\omega)\sin\omega t]\,d\omega=$$
$$=\frac{1}{\pi}\int_{0}^{\infty}S(\omega)\cos[\omega t-\varphi(\omega)]\,d\omega \tag{5}$$

where

$$a(\omega)=\int_{-\infty}^{\infty}f(t_1)\cos\omega t_1\,dt_1;\quad b(\omega)=\int_{-\infty}^{\infty}f(t_1)\sin\omega t_1\,dt_1 \tag{6}$$

$$S(\omega)=\sqrt{[a(\omega)]^2+[b(\omega)]^2};\quad \varphi(\omega)=\tan^{-1}\frac{b(\omega)}{a(\omega)} \tag{7}$$

The relations (6) and (7) show that a non-periodic function, as noted above, has a continuous spectrum. Consequently, the function $f(t)$ can be represented as a set of sinusoidal oscillations of all frequencies, each of these frequencies corresponding to an oscillation of the (infinitesimal) amplitude $S(\omega)\,d\omega/\pi$ and initial phase $\varphi(\omega)$. The quantity $S(\omega)$ is called the *distribution function of the spectrum amplitudes* or *spectral density*. The quantities $a(\omega)$ and $b(\omega)$ are called the cosine and sine components of the spectral density. The quantity $\varphi(\omega)$ is known as the distribution function of initial phases. Formulas (7) define the amplitude and phase spectra of the function $f(t)$.

For even functions the sine component of the spectral density $b(\omega)=0$, and for odd functions the cosine component $a(\omega)=0$.

For instance, the Fourier integral is to be used to represent the even function

$$f(t)=\begin{cases} H & \text{at } |t|<\dfrac{T}{2} \\ 0 & \text{at } |t|>\dfrac{T}{2}\end{cases}$$

which describes a rectangular pulse of duration T and swing H. In this case

$$S(\omega)=|a(\omega)|=2\left|\int_0^{\infty} f(t_1)\cos\omega t_1\,dt_1\right|=$$

$$=2H\left|\int_0^{\frac{T}{2}}\cos\omega t_1\,dt_1\right|=\frac{2H}{\omega}\left|\sin\frac{\omega T}{2}\right|$$

The last expression defines the amplitude spectrum of the function under consideration. Its phase spectrum is given by

$$\varphi(\omega)=\begin{cases} 0 & \text{at } \sin\frac{\omega T}{2}>0 \\ \pi & \text{at } \sin\frac{\omega T}{2}<0 \end{cases}$$

Thus, the function $f(t)$ can be represented by the following integral:

$$f(t)=\frac{2H}{\pi}\int_0^{\infty}\frac{1}{\omega}\left|\sin\frac{\omega T}{2}\right|\cos\omega t\,d\omega$$

Let us now consider, as a second example, the same rectangular pulse with its beginning made to coincide with the origin of time. Then

$$f(t)=\begin{cases} H & \text{at } 0<t<T \\ 0 & \text{at } t<0 \text{ and at } t>T \end{cases}$$

In this case

$$a(\omega)=H\int_0^{T}\cos\omega t_1\,dt_1=\frac{H}{\omega}\sin\omega T$$

$$b(\omega)=H\int_0^{T}\sin\omega t_1\,dt_1=\frac{H}{\omega}(1-\cos\omega T)$$

Hence

$$S(\omega)=\frac{H}{\omega}\sqrt{\sin^2\omega T+(1-\cos\omega T)^2}=\frac{2H}{\omega}\left|\sin\frac{\omega T}{2}\right|$$

which is identical with the result obtained in the first case but the distribution function of the initial phases is different:

$$\varphi(\omega)=\tan^{-1}\frac{1-\cos\omega T}{\sin\omega T}=\frac{\omega T}{2}$$

and the function considered can be consequently represented by the Fourier integral

$$f(t)=\frac{2H}{\pi}\int_{0}^{\infty}\frac{1}{\omega}\left|\sin\frac{\omega T}{2}\right|\cos\omega\left(t-\frac{T}{2}\right)d\omega$$

It is of use to note that there exists a class of non-periodic oscillations with a line spectrum. Such oscillations (and the functions describing them) are known as *almost periodic*. An almost-periodic function has in its spectrum at least two sinusoidal components whose frequency ratio is irrational, for instance, $\omega_2 : \omega_1 = \sqrt{3}$.

A time interval $\theta(\varepsilon)$ termed the *quasi-period* of the almost-periodic function $\Phi(t)$ can be found for any $\varepsilon > 0$, no matter how small. After this time interval the function is repeated for any t with a deviation not over ε, i.e., the inequality

$$|\Phi(t+\theta)-\Phi(t)|<\varepsilon$$

is satisfied.

CHAPTER 2

VIBRATIONS OF LINEAR SYSTEMS WITH CONSTANT PARAMETERS

5. Mechanical Systems

We shall apply the term *system* to a set of objects or elements (real or idealized) which should be isolated in order to facilitate the solution of a problem. A system may comprise, in part or fully, its interactions with the surroundings. The interactions of a mechanical system with the surroundings can be specified as active external factors, which are functions of time (or, in particular cases, constants), and constraints restricting the movement of the system; the constraint equations may be expressed only in terms of coordinates and their derivatives with respect to time (*scleronomous* or *stationary constraints*); they may also contain time in explicit form (*rheonomous* or *nonstationary constraints*).

These constraints and systems with such constraints are termed *holonomic* if the constraint equations contain the coordinates of a system rather than their derivatives with respect to time or differentials of the coordinates; the same term is applied if the constraint equations can be reduced by integration to the form mentioned. If the constraints are not holonomic, their equations are differential equations that are not integrable. Systems which are not holonomic will not be treated in this book.

Constraints are called *ideal* if the work done by their reactions on all displacements they permit is equal to zero. A system is called *autonomous* if the constraints imposed on it are scleronomous and if the system is not subject to variable external forces and its properties (parameters) do not change with time. Otherwise the system is said to be *non-autonomous*. Thus, the differential equations describing the behaviour of an autonomous system do not contain explicit functions of time. In contrast to this, in the differential equations of non-autonomous systems time appears explicitly.

If random processes occur in the system or it is subjected to external random forces, it is called *stochastic*. In the opposite case the system is termed *deterministic*.

The position of a system at a given moment is defined by the values of its coordinates. The state of the system at a given moment is defined by the values of its coordinates and velocities at that time.

Deterministic systems are classified as dynamic or having a history (hereditary). The subsequent behaviour of a dynamic[1] system as well as its behaviour in the past is completely determined if its state is specified at any moment of time. To determine the subsequent behaviour of a system having a history, it is necessary to know, apart from its state at some moment of time, the processes that have occurred in it up to the moment of time in question, i.e., what it has inherited from these processes, besides the conditions of state.

We shall concern ourselves only with dynamic systems. It should be stressed here that properly selected dynamic systems can be used to simulate such characteristic hereditary features, as, for example, elastic after-effect and stress relaxation.

The number of degrees of freedom of the system is usually taken to be the minimum number of independent quantities that must be specified to determine completely the position of the system. These quantities which may have different physical nature are called *generalized coordinates* and can, generally speaking, be arbitrarily selected from a multitude of possible sets. To be more exact, the number of degrees of freedom is equal to half the minimum number of the conditions (generalized coordinates and their derivatives with respect to time), which, when specified, can completely determine the state of the system at some moment.

The order of an ordinary differential equation (or of a set of such equations) describing the behaviour of a system is equal to twice the number of its degrees of freedom when this number is finite. The behaviour of a system having an infinite number of degrees of freedom is described by partial differential equations.

Systems may be *linear* or *nonlinear*. The behaviour of linear systems is described by equations which are linear in coordinates and their derivatives. A system is called *parametric* if the coefficients of coordinates and their derivatives depend on time. Nonlinear systems are described by nonlinear equations.

The interaction forces between the elements of ordinary linear as well as of many nonlinear and parametric mechanical systems may be divided into three groups: *position forces* determined by the coordinates of a system; *dissipative forces* depending on velocities; and *inertia forces* determined by accelerations. A system in which dissipative forces are present is called *dissipative*. Such a system is characterized by energy dissipation determined by the work of the dissipative forces. If there is no dissipation of energy in a system, it is called *conservative*. There exist however non-conservative systems

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[1] Sometimes only systems described by ordinary differential equations, in particular autonomous systems, are referred to as dynamic.

in which the energy is not only dissipated but is also introduced from outside.

Autonomous systems, both linear and nonlinear, may be vibratory or non-vibratory. A vibratory system firstly has at least one position of stable equilibrium, and secondly, when displaced from this position (not too far away) performs free vibrations about it, i.e., vibrates without being acted upon by external forces. Non-vibratory systems have no such property. They may have no position of stable equilibrium. The free motions (in the absence of any forcing factor) of such systems are not vibratory.

6. Free Motions of a Single-Degree-of-Freedom System

Consider a system in a plane as shown in Fig. 7. A rigid body *1* of constant mass m is connected to one end of massless spring *2* whose other end is fastened to a fixed wall *3*. Owing to the presence of guides *4* body *1* can perform only translational rectilinear motions along the axis Ox. The body moves without friction in the guides; neither are there any other resistances. The origin of coordinates O is the position of equilibrium at which the spring is not stressed. The coefficient of stiffness c of spring *2* is constant. Since the system experiences no external forces, its motion is called *free*.

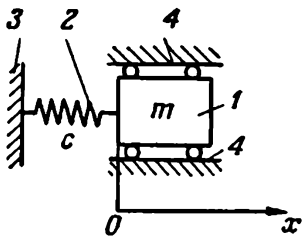

Figure 7

According to Newton's second law, the differential equation of the free motion of the system is

$$m\ddot{x} = S \tag{1}$$

where $\ddot{x}$ = acceleration of body *1*

S = force applied by spring *2* to body *1*.

Forces of this kind directed towards the equilibrium position are called *restoring forces*. They tend, as it were, to restore the position of equilibrium. As the coefficient of stiffness c is constant, the force S is defined by the relation

$$S = -cx \tag{2}$$

where x is the displacement of body *1* from the position of stable equilibrium.

Substituting (2) into Eq. (1), we obtain

$$m\ddot{x} + cx = 0 \tag{3}$$

or

$$\ddot{x} + \omega_0^2 x = 0 \tag{4}$$

where

$$\omega_0 = \sqrt{\frac{c}{m}} \tag{5}$$

As known the general solution of the homogeneous linear differential equation (4) of the second order with constant coefficients is

$$x = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t \tag{6}$$

where t = time

C_1 and C_2 = arbitrary constants which can be determined from the initial conditions:

$$\left.\begin{aligned} \text{at} \quad t = 0 \quad x &= x_0 \\ \dot{x} &= \dot{x}_0 \end{aligned}\right\} \tag{7}$$

Differentiating Eq. (6) with respect to time, we have

$$\dot{x} = -C_1\omega_0 \sin \omega_0 t + C_2\omega_0 \cos \omega_0 t \tag{8}$$

Substituting the initial conditions (7) into expressions (6) and (8) yields two equations. Solving these two equations for the arbitrary constants, we obtain

$$C_1 = x_0; \quad C_2 = \frac{\dot{x}_0}{\omega_0} \tag{9}$$

With these solutions the integral of the differential equation (4) takes the following form:

$$x = x_0 \cos \omega_0 t + \frac{\dot{x}_0}{\omega_0} \sin \omega_0 t \tag{10}$$

It can be rewritten as

$$x = x_a \cos (\omega_0 t - \varphi) \tag{11}$$

where

$$x_a = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2}}; \quad \varphi = \tan^{-1} \frac{\dot{x}_0}{x_0\omega_0} \tag{12}$$

Consequently, the system performs sinusoidal vibrations of amplitude x_a, initial phase $-\varphi$ and angular frequency ω_0. These vibrations are termed *free* or *natural* and the angular frequency ω_0 is called the *natural angular frequency* or *angular frequency of free vibrations*.

It can be seen from Eq. (5) that the natural frequency ω_0 is a function only of the parameters c and m of the system and is independent of the initial conditions and hence of the amplitude. Vibrations whose frequency remains constant at various amplitudes are called *isochronous*. The natural vibrations of a linear system are isochronous.

As follows from Eqs. (12), the amplitude and initial phase of the vibrations are determined, in the general case, by the initial conditions and the parameters of the system. Since the system under consideration is conservative, its free vibrations continue for any length of time at a constant swing.

The natural frequency of a linear system depends on the spring stiffness but is independent of the force of spring tension or compression. This can be illustrated by the following example. Let body *1* (Fig. 8) be suspended from a massless spring *2* in such a way that it can move only in the vertical direction. In distinction to

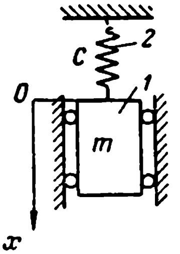

Figure 8

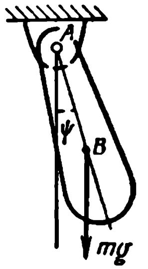

Figure 9

the system in Fig. 7, the gravity force mg is now applied to body *1* (g is the acceleration of a freely falling body). Let the coordinate axis Ox be directed downwards and the origin be placed at the equilibrium position. The equation expressing the static balance of forces is

$$mg = cx_{st} \tag{13}$$

where x_{st} is the static deformation of spring *2* under the action of the weight of body *1*.

The differential equation of motion is of the form

$$m\ddot{x} + c(x + x_{st}) = mg \tag{14}$$

or, on the basis of condition (13),

$$m\ddot{x} + cx = 0$$

which is identical with Eq. (3). Thus, the force of gravity causing additional deformation of the spring does not affect the nature of the vibration of the bo ly about the equilibrium position. The gravity force only displaces he equilibrium position by the amount of the static deformation of the spring.

Comparing Eqs. (5) and (13), we can write

$$x_{st} = \frac{g}{\omega_0^2} \tag{15}$$

Hence, taking into account Eq. (5), Sec. 1, we get

$$x_{st} = \frac{248}{f_0^2}\,[\mathrm{mm}] \tag{16}$$

A classical example of a single-degree-of-freedom vibratory system is the pendulum. Figure 9 shows a pendulum whose fixed hinge axis is at point A and the centre of gravity at point B. In the equilibrium position the line AB is vertical. For the free vibrations of the pendulum in the absence of dissipative resistance forces the differential equation of motion takes the form

$$J\ddot{\psi} + mgl\sin\psi = 0 \tag{17}$$

where J = moment of inertia of the pendulum with respect to the hinge axis A
m = mass of the pendulum
l = distance AB from the hinge axis to the centre of gravity
ψ = angle of displacement of the pendulum from the equilibrium position.

Equation (17) is nonlinear. With sufficiently small angles ψ, however, it can be rewritten as follows:

$$J\ddot{\psi} + mgl\psi = 0 \tag{18}$$

Equation (18) yields the natural frequency of the small oscillationsof the pendulum

$$\omega_0 = \sqrt{\frac{mgl}{J}} \tag{19}$$

The so-called mathematical pendulum has all its mass concentrated at the centre of gravity. For this pendulum

$$J_m = ml_m^2$$

the subscript m indicating that the pendulum meant is the mathematical one. Hence, for this pendulum

$$\omega_0 = \sqrt{\frac{g}{l_m}} \tag{20}$$

Relation (19) can be used to determine experimentally the acceleration of gravity from the oscillation period of the pendulum. Formula (15) is not suitable for this purpose as the period of vibration of a spring-suspended mass is independent of theacceleration of gravity. The acceleration determines only the static deformation of the spring.

Let the pendulum shown in Fig. 9 have, instead of a free hinge, an elastic one whose angular stiffness s is constant. In the equilibrium position the pendulum is displaced from the vertical by the angle α. Measuring the displacement angle ψ of the pendulum

from the equilibrium position, we can write the differential equation of the oscillations as follows:

$$J\ddot{\psi}+mgl\sin(\alpha+\psi)+s(\psi-\psi_{st})=0 \tag{21}$$

where ψ_{st} is the static angular deformation of the elastic hinge under the action of the gravity-force moment; it is determined by the condition

$$s\psi_{st}=mgl\sin\alpha \tag{22}$$

This relation is also valid at $\alpha = \pi/2$.

Consider now the free motion of the dissipative system illustrated in Fig. 10. This system differs from the one shown in Fig. 7

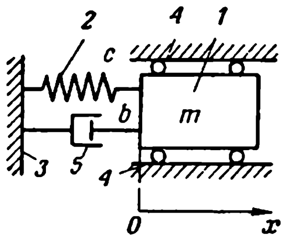

Figure 10

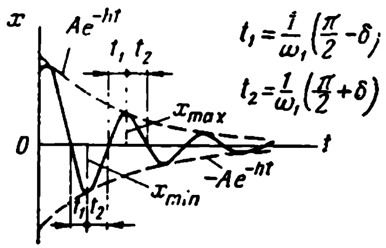

Figure 11

Hence, for small oscillations we obtain the differential equation

$$J\ddot{\psi}+(mgl\cos\alpha+s)\psi=0 \tag{23}$$

The natural frequency of oscillations

$$\omega_0=\sqrt{\frac{mgl\cos\alpha+s}{J}} \tag{24}$$

In the special case of $l=0$, i.e., when the hinge axis passes through the centre of gravity

$$\omega_0=\sqrt{\frac{s}{J}} \tag{25}$$

in that it has a damper or cataract *5* which dissipates energy. The dissipative force B, i.e., the force opposing the movement of body *1* and applied to it by the damper is directed against the direction of the velocity. In our linear diagram this force is proportional to the velocity, i.e.,

$$B=-b\dot{x} \tag{26}$$

where b is the resistance or the coefficient of resistance.[1]

[1] The term *resistance* seems more appropriate. It corresponds to the electrical resistance. Moreover, by using it we preclude the frequent confusion of the coefficient of resistance with the damping coefficient.

The differential equation of motion takes the form

$$m\ddot{x}+b\dot{x}+cx=0 \tag{27}$$

or

$$\ddot{x}+2h\dot{x}+\omega_0^2x=0 \tag{28}$$

where the damping coefficient is

$$h=\frac{b}{2m} \tag{29}$$

and ω_0 is defined by formula (5).

It is well known that the general solution of Eq. (28) and, consequently, the nature of the motion described by the equation depends on the ratio of h to ω_0. Consider three possible cases.

If $h < \omega_0$, the general solution of Eq. (28) is of the form

$$x=e^{-ht}(C_1\cos\omega_1 t+C_2\sin\omega_1 t) \tag{30}$$

where C_1, C_2 are arbitrary constants;

$$\omega_1=\sqrt{\omega_0^2-h^2} \tag{31}$$

For initial conditions (7)

$$C_1=x_0; \qquad C_2=\frac{x_0h+\dot{x}_0}{\omega_1} \tag{32}$$

Expression (30) can also be rewritten as

$$x=Ae^{-ht}\cos(\omega_1 t-\varphi) \tag{33}$$

where

$$\left.\begin{aligned} A&=\sqrt{x_0^2+\frac{(x_0h+\dot{x}_0)^2}{\omega_1^2}}\\ \varphi&=\tan^{-1}\frac{x_0h+\dot{x}_0}{x_0\omega_1}\end{aligned}\right\} \tag{34}$$

Formula (33) shows that the free motion in this case is a decaying vibration whose time-history is illustrated in Fig. 11. The vibrations continue to infinity and their swing value tends asymptotically to zero. Such terms as the period, frequency and amplitude of vibration, strictly speaking, are not applicable to this kind of vibrations.

The envelopes $\pm Ae^{-ht}$ are tangent to the curve of decaying free vibrations at points where $\cos(\omega_1 t-\varphi)=\pm 1$.

Extreme x values are ahead of the extremes of $\cos(\omega_1 t-\varphi)$ by the time interval

$$\Delta t_{ext}=\frac{\delta}{\omega_1} \tag{35}$$

where δ is the so-called *loss angle* defined by the relation

$$\tan\delta = \frac{h}{\omega_1} \tag{36}$$

The extreme values

$$|x_{ext}| = A\frac{\omega_1}{\omega_0}e^{-ht_{ext}} \tag{37}$$

where

$$t_{ext} = \frac{1}{\omega_1}(\varphi - \delta + \pi k), \qquad (k = 0, 1, 2, \ldots) \tag{38}$$

The maxima correspond to even k values, the minima to odd k values.

The time interval between two neighbouring maxima or minima or between two nearest zero points through which the vibrating body passes in the same direction is given by

$$T_1 = \frac{2\pi}{\omega_1} \tag{39}$$

The velocity and acceleration of vibration are obtained by differentiating Eq. (33):

$$\left.\begin{aligned} \dot{x} &= -A\omega_0 e^{-ht}\sin(\omega_1 t - \varphi + \delta) \\ \ddot{x} &= -A\omega_0^2 e^{-ht}\cos(\omega_1 t - \varphi + 2\delta) \end{aligned}\right\} \tag{40}$$

A distinctive feature of the vibrations is the constant ratio of two successive maxima or minima:

$$\frac{(x_{ext})_k}{(x_{ext})_{k+2}} = e^{\frac{2\pi h}{\omega_1}} \tag{41}$$

The natural logarithm of this ratio

$$\vartheta = \ln\left[\frac{(x_{ext})_k}{(x_{ext})_{k+2}}\right] = \frac{2\pi h}{\omega_1} = hT_1 = 2\pi\tan\delta \tag{42}$$

is called the *logarithmic decrement* of vibrations.

The logarithmic decrement, like the tangent of the loss angle, is a suitable dimensionless parameter characterizing the dissipation of energy in a system.

Other dimensionless quantities are also often used for this purpose, for example, the damping ratio[1].

$$\beta = \frac{h}{\omega_0} = \sin\delta \tag{43}$$

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[1] This is basically an expression of the damping coefficient if its critical value at which the system is no longer vibratory is taken as the unit of measurement.

and the quality factor

$$Q = \frac{1}{2\beta} = \frac{\omega_0}{2h} \tag{44}$$

The absorption coefficient is also sometimes used as a dimensionless parameter determining energy dissipation. Unfortunately, the term is interpreted differently by different authors. Four of these interpretations of the term *absorption coefficient* are given below and denoted by the symbols Ψ, Ψ_1, Ψ_2, Ψ_3.

The absorption coefficient Ψ is defined by the expression

$$\Psi = -\int_{E_k}^{E_{k+2}} \frac{dE}{E} = \ln \frac{E_k}{E_{k+2}}$$

where E = running value of the system energy

E_k and E_{k+2} = values of the (potential) energy for the kth and $(k+2)$nd extreme values of displacement x.

Since

$$E_k = \frac{1}{2} c\,(x_{ext})_k^2, \qquad E_{k+2} = \frac{1}{2} c\,(x_{ext})_{k+2}^2$$

we obtain from formula (42)

$$\Psi = 2\vartheta$$

The absorption coefficients Ψ_1, Ψ_2, Ψ_3 are the ratios of the energy dissipated during one cycle of free vibration to the arithmetic mean of energy values at the beginning and end of the cycle, and to the initial and final energy values, respectively. These ratios as well as the exact dependences of the absorption coefficients on the logarithmic decrement ϑ and their expansion in power series of ϑ are:

$$\Psi_1 = \frac{2(E_k - E_{k+2})}{E_k + E_{k+2}} = \frac{2(e^{2\vartheta} - 1)}{e^{2\vartheta} + 1} = 2\vartheta - \frac{2}{3}\vartheta^3 - \dots$$

$$\Psi_2 = \frac{E_k - E_{k+2}}{E_k} = 1 - e^{-2\vartheta} = 2\vartheta - 2\vartheta^2 + \frac{4}{3}\vartheta^3 - \dots$$

$$\Psi_3 = \frac{E_k - E_{k+2}}{E_{k+2}} = e^{2\vartheta} - 1 = 2\vartheta + 2\vartheta^2 + \frac{4}{3}\vartheta^3 + \dots$$

These formulas show that Ψ_1 is the coefficient that gives the nearest approach to Ψ, differing from it by a small term of the third order in ϑ. The quantities Ψ_2 and Ψ_3 differ from Ψ by small terms of the second order in ϑ.

Conversion formulas for the dimensionless parameters β, Q, δ, and ϑ are given in Table 1.

It is readily seen that the logarithmic decrement and the loss angle have meaning, i.e., are expressed by real numbers, only when $h < \omega_0$ (to be more exact, when $h \leqslant \omega_0$ for the loss angle), while

TABLE 1

	β	Q	ϑ	δ
$\beta=$	β	$\frac{1}{2Q}$	$\frac{1}{\sqrt{\frac{4\pi^2}{\vartheta^2}+1}}$	$\sin\delta$
$Q=$	$\frac{1}{2\beta}$	Q	$\frac{1}{2}\sqrt{\frac{4\pi^2}{\vartheta^2}+1}$	$\frac{1}{2}\operatorname{cosec}\delta$
$\vartheta=$	$\frac{2\pi}{\sqrt{\frac{1}{\beta^2}-1}}$	$\frac{2\pi}{\sqrt{4Q^2-1}}$	ϑ	$2\pi\tan\delta$
$\delta=$	$\sin^{-1}\beta$	$\sin^{-1}\frac{1}{2Q}$	$\tan^{-1}\frac{\vartheta}{2\pi}$	δ

the damping ratio and quality factor may be used for any ratio of h to ω_0.

When $h > \omega_0$, the general solution of Eq. (28) can be presented in the form

$$x = e^{-ht}(C_1 e^{\nu t} + C_2 e^{-\nu t})$$

or

$$x = e^{-ht}(C_1' \cosh \nu t + C_2' \sinh \nu t) \tag{45}$$

where

$$\nu = \sqrt{h^2 - \omega_0^2} \tag{46}$$

In this case the free motion of the system is no longer oscillatory Since $h > \nu$, $\lim\limits_{t\to\infty} x = 0$, i.e., the system, in the final analysis, asymptotically approaches the equilibrium position.

From the initial conditions (7) we obtain

$$C_1 = \frac{1}{2\nu}[x_0(h+\nu) + \dot{x}_0]$$

$$C_2 = -\frac{1}{2\nu}[x_0(h-\nu) + \dot{x}_0] \tag{47}$$

Depending on the relation between the initial conditions and the parameters of the system its motion can take one of the three

forms described below. In discussing the three cases use will be made of two dimensionless parameters

$$\rho=\frac{\dot{x}_0}{x_0(h+\nu)} \quad \text{and} \quad \chi=\frac{h-\nu}{h+\nu} \tag{48}$$

If at the initial moment the body *1* (see Fig. 10) has a comparatively high velocity in the direction of the equilibrium position so that $\rho < -1$, it will then pass through the equilibrium position at the moment

$$t_0=\frac{1}{2\nu}\ln\frac{\chi+\rho}{1+\rho} \tag{49}$$

The displacement x then reaches its extreme value at the moment

$$t_m=\frac{1}{2\nu}\ln\frac{1+\frac{\rho}{\chi}}{1+\rho} \tag{50}$$

and from this moment on it changes monotonically, asymptotically approaching zero. This is illustrated by curves *1* and *2* in Fig. 12*a*.

If at the initial moment body *1* (see Fig. 10) is moving away from the equilibrium position so that $\rho > 0$, then x reaches its extreme value at the moment given by expression (50) and from this moment on changes monotonically, asymptotically approaching zero. Curves *3* and *4* in Fig. 12*b* represent this motion.

If body *1* (see Fig. 10) is moving, at the initial moment, towards the equilibrium position at a comparatively low velocity so that $-1 \leqslant \rho \leqslant 0$, then x changes monotonically from the very beginning, asymptotically approaching zero as shown by curves *5* and *6* in Fig. 12*c*. The motion, specifically at $\rho = -1$ or $\rho = -\chi$, is such that the velocity remains proportional to the displacement from the equilibrium position at all times.

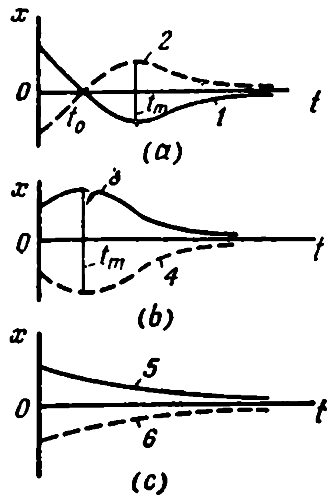


Figure 12

And, finally, consider the intermediate case when $h = \omega_0$. The general solution of Eq. (28) in this case can be presented as

$$x=e^{-ht}(C_1+C_2t) \tag{51}$$

This case does not differ qualitatively from the preceding one. Let us use the notation

$$\mu=\frac{\dot{x}_0}{x_0h} \tag{52}$$

The motion of the type represented by curves *1* and *2* in Fig. 12*a* will be realized with $\mu < -1$ and the body will pass through the equilibrium position at the moment

$$t_0 = -\frac{1}{h(1+\mu)} \tag{53}$$

and the extreme value of x will be attained at the moment

$$t_m = \frac{1}{h\left(1+\frac{1}{\mu}\right)} \tag{54}$$

The motion of the type represented by curves *3* and *4* (Fig. 12*b*) will be realized when $\mu > 0$. The moment at which the extreme value is reached is determined by formula (54).

The motion of the type shown by curves *5* and *6* (Fig. 12*c*) corresponds to $-1 \leqslant \mu \leqslant 0$. In the particular case when $\mu = -1$ the velocity will remain proportional to the displacement from the equilibrium position all the time.

Thus, free motion is a vibration when $h < \omega_0$ ($\beta < 1$, or $Q > \frac{1}{2}$, or $\vartheta < \infty$, or $\delta < \pi/2$) and is no longer a vibration at $h \geqslant \omega_0$. The value of damping at which the motion loses its vibratory character is called *critical*:

$$h_{cr} = \omega_0 \tag{55}$$

Of practical interest is the case when spring *2* in the diagram in Fig. 10 is absent. The differential equation of the free motion can then be written down thus:

$$m\ddot{x} + b\dot{x} = 0 \tag{56}$$

or

$$\ddot{x} + 2h\dot{x} = 0 \tag{57}$$

The general solution of the last equation is

$$x = C_1 + C_2 e^{-2ht} \tag{58}$$

which is the limiting value of (45) at $\nu = h$. For the initial values (7) we have

$$C_1 = x_0 + \frac{\dot{x}_0}{2h}, \qquad C_2 = -\frac{\dot{x}_0}{2h} \tag{59}$$

whence

$$x = x_0 + \frac{\dot{x}_0}{2h}(1 - e^{-2ht}) \tag{60}$$

$$\dot{x} = \dot{x}_0 e^{-2ht} \tag{61}$$

This system may be in equilibrium in any position[1]. It moves monotonically, asymptotically approaching the coordinate value $x_0 + \dot{x}_0/2h$. Figure 13a and b shows the time-histories of displacement and velocity, respectively. No matter how small is the initial damping ratio of the system in Fig. 10, it can be made greater than unity by reducing the mass of body 1. With sufficient reduction of the mass the first term on the left-hand side of Eq. (28) becomes

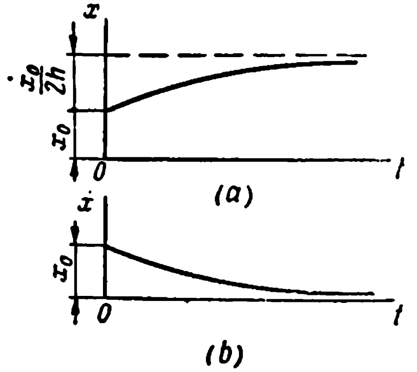


Figure 13

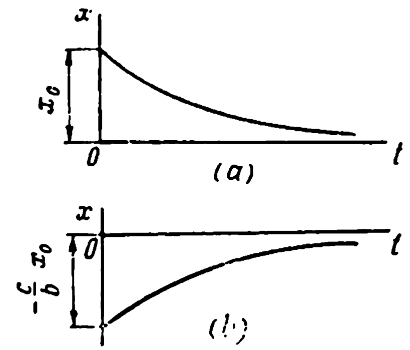


Figure 14

mall compared with the rest and it may prove to be reasonable to neglect it and write the equation as follows:

$$b\dot{x} + cx = 0 \tag{62}$$

or

$$\dot{x} + \frac{c}{b}x = 0 \tag{63}$$

This is a differential equation of the first order; consequently, the degree of freedom of the system is 1/2. The general solution is

$$x = C_1 e^{-\frac{c}{b}t} \tag{64}$$

From the initial condition $x = x_0$ at $t = 0$ we obtain

$$C_1 = x_0 \tag{65}$$

Hence

$$x = x_0 e^{-\frac{c}{b}t} \tag{66}$$

The initial velocity of the system cannot be chosen arbitrarily. It is determined unambiguously from the equation

$$\dot{x} = -x_0 \frac{c}{b} e^{-\frac{c}{b}t} = -\frac{c}{b}x \tag{67}$$

i.e., at $t = 0$

$$\dot{x}_0 = -\frac{c}{b}x_0 \tag{68}$$

[1] This state is called *indifferent equilibrium* (see Section 10).

If a different initial velocity is imparted to the system, it will instantly assume the value given by formula (68). Time-histories of the displacement and velocity of the system are shown in Fig. 14*a* and *b*. Both the displacement and the velocity change monotonically, asymptotically approaching zero. Formula (67) shows that the velocity remains proportional to the displacement at all times.

The term *time constant* (or *relaxation time*) is often used in the literature; it is defined as the time t_c during which the free-motion velocity or its curve envelope ordinate diminishes by a factor of $1/e$ ($e = 2.71828\ldots$ is the base of natural logarithms). The term time constant has meaning, i.e., is really a constant, only when applied to systems whose velocity of free motion (or the ordinate of the envelope of the velocity curve) is proportional to the displacement (or to the ordinate of the envelope of the displacement curve)

TABLE 2

Differential equation	Relation between parameters	Special conditions	t_c
$\ddot{x}+2h\dot{x}+\omega_0^2x=0$ $\left(h=\frac{b}{2m},\ \omega_0=\sqrt{\frac{c}{m}}\right)$	$h<\omega_0$	—	$t_c=\frac{1}{h}=\frac{2m}{b}$
	$h=\omega_0$	$\frac{\dot{x}_0}{x_0h}=-1$	$t_c=\frac{1}{h}=\frac{2m}{b}$
	$h>\omega_0$	$\frac{\dot{x}_0}{x_0(h+\nu)}=-1$ $(\nu=\sqrt{h^2-\omega_0^2})$	$t_c=\frac{1}{h+\nu}=\frac{2m}{b}\times\frac{1}{1+\sqrt{1-\frac{4cm}{b^2}}}$
		$\frac{\dot{x}_0}{x_0(h-\nu)}=-1$	$t_c=\frac{1}{h-\nu}=\frac{2m}{b}\times\frac{1}{1-\sqrt{1-\frac{4cm}{b^2}}}$
$\ddot{x}+2h\dot{x}=0$	—	—	$t_c=\frac{1}{2h}=\frac{m}{b}$
$\dot{x}+\frac{c}{b}x=0$	—	—	$t_c=\frac{b}{c}$

reckoned from the position that the system will assume at $t = \infty$. The values of the time constant for various cases are listed in Table 2.

7. Forced Vibrations of a Single-Degree-of-Freedom System Excited by a Sinusoidal Force

Unlike the free motions considered in the preceding section, forced motions are generated by external factors acting on the system, which are represented by functions of time. If the motion is caused by variable external forces, it is usually spoken of as force or dynamically excited. If the motion of the system is caused by an externally specified motion of its elements or points, the motion is said to be *kinematically excited.*

We shall apply the term *exciting force* to the variable force causing the motion [1].

Consider the case of a sinusoidal force applied to body *1*, an element of a conservative system (see Fig. 7),

$$F = F_a \cos \omega t \tag{1}$$

where F_a = amplitude of the exciting force
ω = angular frequency of force oscillations.

The differential equation of the motion of the system can be written in this case as:

$$m\ddot{x} + cx = F_a \cos \omega t \tag{2}$$

or

$$\ddot{x} + \omega_0^2 x = \frac{F_a}{m} \cos \omega t \tag{3}$$

where ω_0 is defined by formula (5), Sec. 6.

The general solution of the nonhomogeneous linear equation (3) may be represented as the sum of the general solution of the corresponding homogeneous linear equation (4), Sec. 6, and a particular integral X of Eq. (3):

$$x = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + X \tag{4}$$

We shall seek the particular integral in the form

$$X = x_a \cos \omega t \tag{5}$$

Substituting expression (5) into the left-hand side of Eq. (3), we obtain

$$x_a = \frac{F_a}{m(\omega_0^2 - \omega^2)} \quad (6$$

[1] The terms *disturbing force* and *forcing factor* are also used.

and, consequently,

$$x = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_a}{m(\omega_0^2 - \omega^2)} \cos \omega t \tag{7}$$

whence

$$\dot{x} = -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t - \frac{F_a \omega}{m(\omega_0^2 - \omega^2)} \sin \omega t \tag{8}$$

Substituting the initial conditions (7), Sec. 6, into Eqs. (7) and (8), one has

$$C_1 = x_0 - \frac{F_a}{m(\omega_0^2 - \omega^2)}, \qquad C_2 = \frac{\dot{x}_0}{\omega_0} \tag{9}$$

The general solution (4) can now be presented in the following form:

$$x = x_0 \cos \omega_0 t + \frac{\dot{x}_0}{\omega_0} \sin \omega_0 t - \frac{F_a}{m(\omega_0^2 - \omega^2)} \cos \omega_0 t + \frac{F_a}{m(\omega_0^2 - \omega^2)} \cos \omega t \tag{10}$$

The first three terms on the right-hand side of Eq. (10) are vibrations at the natural frequency of the system and the last term represents vibrations at the exciting force frequency. The vibrations at

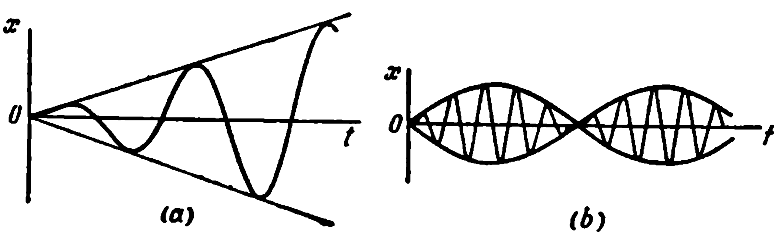


Figure 15

the natural frequency consist of two parts of different origin. The first two terms depend on the initial conditions and are independent of the exciting force; these vibrations had existed before the exciting force was applied; let us call them the initial natural vibrations. The third term is a function of the exciting force and is independent of the initial conditions; the vibrations it represents are caused by the exciting force F applied to the system that had been free at the moment $t = 0$; let us call them the excited natural vibrations.

The ideal system under consideration will perform natural vibrations at constant amplitude for an infinite time. In real systems the dissipative forces will make the natural vibrations die out, as a result of which the system, in the final analysis, will perform forced vibrations:

$$x = \frac{F_a}{m(\omega_0^2 - \omega^2)} \cos \omega t \tag{11}$$

It can readily be seen from expression (11) that at $\omega < \omega_0$ the forced vibrations are in phase and at $\omega > \omega_0$ in opposite phase with the exciting force. The amplitude of the vibrations increases as ω approaches ω_0 and becomes infinitely large at $\omega = \omega_0$. However, as will be presently seen, the infinitely large amplitude is attained after an infinitely long time. Let us rewrite expression (10) using zero initial conditions in order to simplify it:

$$x = \frac{F_a (\cos \omega t - \cos \omega_0 t)}{m (\omega_0^2 - \omega^2)} \tag{12}$$

The limit value of the right-hand side of Eq. (12) takes the indeterminate form $\frac{0}{0}$ when $\omega \to \omega_0$. Let us find the value of (12) using L'Hôpitale's rule. Differentiating the numerator and denominator of Eq. (12) with respect to ω, we get

$$\lim_{\omega \to \omega_0} \frac{F_a (\cos \omega t - \cos \omega_0 t)}{m (\omega_0^2 - \omega^2)} = \lim_{\omega \to \omega_0} \frac{F_a t \sin \omega t}{2m\omega} = \frac{F_a}{2m\omega_0} t \sin \omega_0 t \tag{13}$$

Hence

$$x = \frac{F_a}{2m\omega_0} t \sin \omega_0 t = \frac{F_a}{2m\omega_0} t \cos \left(\omega_0 t - \frac{\pi}{2}\right) \tag{14}$$

It then follows that when $\omega = \omega_0$ the forced vibrations of the system cease to be stationary: they become more and more severe. The "amplitude" of the vibrations is now proportional to the time elapsed. The time-history of the vibrations is presented in Fig. 15*a*. The vibrations (to be more precise, the sinusoidal factor) lag in phase with respect to the exciting force by the angle $\pi/2$.

The vibration phenomenon characterized by increased amplitudes when the frequency of the exciting force exactly coincides with that of free vibration of the system is called *resonance* (see Sec. 13). Vibrations at $\omega = \omega_0$ are called *resonant*, at $\omega < \omega_0$ *preresonant*, and at $\omega > \omega_0$ *postresonant vibrations*.

It is important to note that in the nonresonance case after application of the exciting force the half-swing of the vibrations at zero initial conditions can attain the double value of the stationary vibration amplitude determined by relation (11). This follows from expression (12) which can be rewritten in the following form:

$$x = \frac{2F_a}{m (\omega_0^2 - \omega^2)} \sin \frac{\omega_0 - \omega}{2} t \sin \frac{\omega_0 + \omega}{2} t \tag{15}$$

The time-history of the vibrations is shown in Fig. 15*b*. The maximum value of the half-swing is attained near the moment of time at which $\left| \sin \frac{\omega_0 - \omega}{2} t \right| = 1$; hence $t = \frac{\pi}{|\omega_0 - \omega|}$, i.e., the maximum is reached the later the nearer ω approaches ω_0 (see Fig. 15*b*, in which ω is near enough to ω_0 and the vibrations are of the nature

of beats). Figure 16*a* shows x_a versus ω and Fig. 16*b*, the lag of the vibration phase with respect to the phase of the exciting force. These curves are known as the *amplitude* and *phase response* curves or characteristics. The former is sometimes called the *resonance curve.*

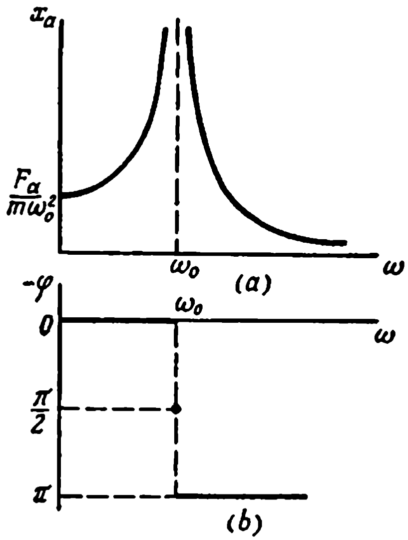


Figure 16

Some characteristics of forced vibrations are presented in Table 3.

We now turn to the case in which the sinusoidal force given by expression (1) is applied to body *1* of the dissipative system depicted in Fig. 10. The differential equation of motion is

$$m\ddot{x}+b\dot{x}+cx=F_a\cos\omega t \qquad (16)$$

Hence

$$\ddot{x}+2h\dot{x}+\omega_0^2x=\frac{F_a}{m}\cos\omega t \qquad (17)$$

where ω_0 and h are defined by formulas (5) and (29), Sec. 6.

For the sake of definiteness let us assume that $h<\omega_0$. From ex-

TABLE 3

Forced vibrations	Preresonant $\omega<\omega_0$	Resonant $\omega=\omega_0$	Postresonant $\omega>\omega_0$
Character of vibration	Steady-state	Increasing	Steady-state
Displacement amplitude	$\frac{F_a}{m(\omega_0^2-\omega^2)}$	$\frac{F_a}{2m\omega_0}t$	$\frac{F_a}{m(\omega^2-\omega_0^2)}$
Phase with respect to phase of exciting force	0	$-\frac{\pi}{2}$	$-\pi$
Maximum possible half-swing at zero initial conditions	$\frac{2F_a}{m(\omega_0^2-\omega^2)}$	∞	$\frac{2F_a}{m(\omega^2-\omega_0^2)}$
Approximate time when the maximum half-swing of vibrations is attained	$\frac{\pi}{\omega_0-\omega}$	∞	$\frac{\pi}{\omega-\omega_0}$

pression (30), Sec. 6, we obtain now the general solution of the inear nonhomogeneous equation (17):

$$x = e^{-ht}(C_1 \cos \omega_1 t + C_2 \sin \omega_1 t) + X \tag{18}$$

whose particular integral will be sought in the form

$$X = x_a \cos(\omega t - \varphi) \tag{19}$$

Substituting expression (19) into Eq. (17), we find

$$x_a = \frac{F_a}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}} \tag{20}$$

$$\varphi = \tan^{-1} \frac{2h\omega}{\omega_0^2 - \omega^2} \tag{21}$$

Differentiating Eq. (18), one has

$$\dot{x} = e^{-ht}[-C_1(h \cos \omega_1 t + \omega_1 \sin \omega_1 t) + \\ + C_2(\omega_1 \cos \omega_1 t - h \sin \omega_1 t)] + \dot{X} \tag{22}$$

Substituting the initial conditions (7), Sec. 6, into Eqs. (18) and (22), we determine the arbitrary constants C_1 and C_2. Inserting these constants into Eq. (18) and using formulas (19) and (20), we obtain the solution of differential equation (17) in the following form:

$$x = e^{-ht}\left(x_0 \cos \omega_1 t + \frac{x_0 h + \dot{x}_0}{\omega_1} \sin \omega_1 t\right) - \\ - \frac{F_a e^{-ht}}{m[(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2]}\left[(\omega_0^2 - \omega^2) \cos \omega_1 t + \frac{h}{\omega_1}(\omega_0^2 + \omega^2) \sin \omega_1 t\right] + \\ + \frac{F_a}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}} \cos(\omega t - \varphi) \tag{23}$$

The first term on the right-hand side of Eq. (23) represents the initial natural vibrations determined by the initial conditions and independent of the exciting force; the second term is the excited natural vibrations determined by the exciting force and independent of the initial conditions; the third term expresses the forced vibrations. Natural vibrations, whatever their origin, die out with time as evidenced by the factor e^{-ht} and only steady-state forced vibrations ultimately remain:

$$x = \frac{F_a}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}} \cos(\omega t - \varphi) \tag{24}$$

The above holds also in the case when $h \geqslant \omega_0$ which differs from the one discussed above in that the natural motion is not vibratory.

As distinct from a conservative system, the forced vibrations of a dissipative system are stationary at any frequency of the exciting force.

Having determined the extreme values of expression (20) for the amplitude of forced vibrations, we find that the maximum is attained not at $\omega=\omega_0$, as in the case of the conservative system, but at the resonant frequency

$$\omega_m=\sqrt{\omega_0^2-2h^2}<\omega_0 \tag{25}$$

This maximum value of resonance is given by the expression

$$x_{a\,max}=\frac{F_a}{2mh\sqrt{\omega_0^2-h^2}} \tag{26}$$

Expression (25) shows that the resonance conditions for the vibration displacement in the given system are possible only when $h<\omega_0/\sqrt{2}$. As can be seen from Eq. (26), the quantity $x_{a\,max}$ in the interval $0<h<\omega_0/\sqrt{2}$ diminishes with increase of h.

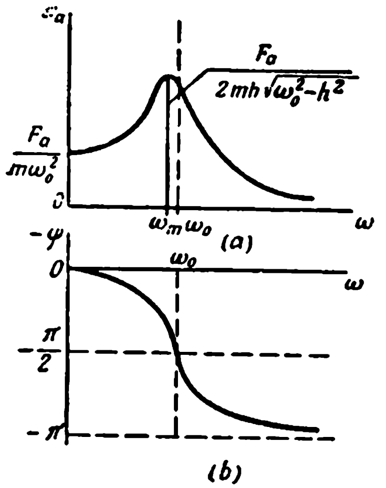


Figure 17

It follows from formula (21) that the displacement phase lags behind the phase of the exciting force by an angle of $0<\varphi<\pi/2$ at $\omega<\omega_0$, by the angle $\pi/2$ at $\omega=\omega_0$, and by the angle $\pi/2<\varphi<\pi$ at $\omega>\omega_0$.

Figure 17*a* illustrates an approximated displacement response curve of a dissipative system ($h<\omega_0/\sqrt{2}$) and Fig. 17*b* is the phase response curve.

Some of the results obtained are given in Table 4.

If the signs in Eq. (16) are reversed, then the first term on the left-hand side will represent the so-called inertia force, the second the dissipative force and the third the restoring force. The restoring force is always directed against the inertia force. At $\omega<\omega_0$ the restoring force is greater than the inertia force; at $\omega>\omega_0$ the opposite is true, the inertia force is greater than the restoring force. At $\omega=\omega_0$ the two forces balance each other completely and therefore in a conservative system the force exciting sinusoidal vibrations is not compensated at all and the vibrations must inevitably increase (resonance). In a dissipative system the exciting force is compensated by the dissipative force.

The use of complex quantities offers certain advantages in the study of steady-state forced vibrations of linear (in particular, complicated) systems. The exciting force is taken in the form

$$F=F_a e^{i\omega t} \tag{27}$$

instead of using expression (1).

TABLE 4

Characteristic	Formula	Necessary condition
Amplitude of forced vibrations (displacement)	$x_a = \frac{F_a}{m\sqrt{(\omega_0^2-\omega^2)^2+4h^2\omega^2}}$	—
Phase of forced vibrations with respect to phase of exciting force	$0 > -\varphi > -\frac{\pi}{2}$ $-\varphi = -\frac{\pi}{2}$ $-\frac{\pi}{2} > -\varphi > -\pi$	$\omega < \omega_0$ $\omega = \omega_0$ $\omega > \omega_0$
Displacement amplitude at resonance	$x_{a\,max} = \frac{F_a}{2mh\sqrt{\omega_0^2-h^2}}$	$h < \frac{\omega_0}{\sqrt{2}}$
Resonant displacement frequency	$\omega_m = \sqrt{\omega_0^2 - 2h^2}$	
Natural frequency	$\omega_1 = \sqrt{\omega_0^2 - h^2}$	$h < \omega_0$

The differential equation of motion of a system having one degree of freedom is then

$$m\ddot{x} + b\dot{x} + cx = F_a e^{i\omega t} \tag{28}$$

or

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = \frac{F_a}{m} e^{i\omega t} \tag{29}$$

Note that by using complex quantities consisting of two terms, the real and the imaginary, we apply the principle of superposition and, consequently, this approach is applicable only to linear systems.

We assume the particular integral corresponding to steady-state forced vibrations to have the form

$$x = Ae^{i\omega t} \tag{30}$$

where A is the complex amplitude of vibration displacement.

Substituting Eq. (30) into the left-hand side of Eq. (29), we obtain

$$A = \frac{F_a}{m(\omega_0^2 - \omega^2 + 2h\omega i)} \tag{31}$$

whence, adopting the notation

$$A = x_a e^{-i\varphi} \tag{32}$$

we get

$$x = x_a e^{i(\omega t - \varphi)} \tag{33}$$

where x_a and φ are given by formulas (20) and (21).

Inserting the integral (30) into the basic equation (28), we have

$$A = \frac{F_a}{c - m\omega^2 + ib\omega} \tag{34}$$

The denominator in (34)

$$C_c = c - m\omega^2 + ib\omega \tag{35}$$

is called the *dynamic* or *complex stiffness* of the system. Obviously

$$x_a = \frac{F_a}{|C_c|} \tag{36}$$

i.e., the amplitude of a forced vibration is equal to the ratio of the amplitude of the exciting force to the modulus of dynamic stiffness.

Using the following notation for the complex amplitudes of vibration velocity and vibration acceleration

$$\dot{A} \equiv i\omega A \text{ and } \ddot{A} \equiv -\omega^2 A \tag{37}$$

we can write

$$\dot{A} = \frac{F_a}{b + i\left(m\omega - \frac{c}{\omega}\right)} \tag{38}$$

where the denominator of the right-hand side

$$b_c = b + i\left(m\omega - \frac{c}{\omega}\right) \tag{39}$$

is called *impedance*, or *mechanical impedance*, or *complex resistance* of the system. Hence

$$\dot{x}_a = \frac{F_a}{|b_c|} \tag{40}$$

i.e., the amplitude of vibration velocity is equal to the ratio of the amplitude of the exciting force to the modulus of impedance. Expression (40) is the mechanical analog of Ohm's law determining the current in a circuit with complex resistance.

Further, we obtain

$$\ddot{A} = \frac{F_a}{m - \frac{c}{\omega^2} - i\frac{b}{\omega}} \tag{41}$$

where the denominator of the right-hand side may be called the *complex mass* m_c. Hence

$$\ddot{x}_a = \frac{F_a}{|m_c|} \tag{42}$$

i.e., the amplitude of vibration acceleration is equal to the ratio of the amplitude of the exciting force to the modulus of the complex mass.

Note that when $\omega = \omega_0$ the dynamic stiffness and the complex mass are imaginary quantities and the impedance is a real quantity.

8. Centrifugal and Kinematic Excitation of Vibrations and a System with Positive Motion of Mass Element

We have so far discussed forced vibrations under the action of a sinusoidal force whose amplitude is independent of the vibration frequency. The case when the amplitude of the exciting force is proportional to the square of the vibration frequency is of great importance in practice. This is the case one has to deal with if the vibrations are centrifugally excited. Figure 18*a* illustrates a system comprising a body *1* of mass m_1 (whose motion is limited by ideal guides *4*), a linear spring *2* of stiffness c and a linear damper *5* whose resistance coefficient is b; the spring and damper connect body *1* with the fixed stand *3*. The vibrations of body *1* are excited by

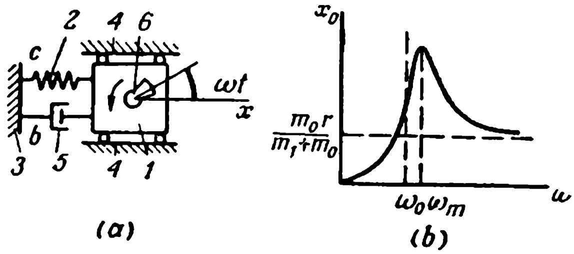


Figure 18

the rotation of unbalanced rotor *6* (unbalance) about the axis rigidly fixed to body *1*. The unbalanced mass *6* rotates at a constant angular velocity ω; its mass is m_0 and it is eccentric with respect to the axis by r (r is the distance from the axis of rotation to the centre of gravity of the unbalanced mass). The rotation of the unbalanced mass generates the centrifugal force

$$F_a = m_0 r \omega^2 \tag{1}$$

whose projection onto the horizontal axis is the exciting force

$$F = F_a \cos \omega t \tag{2}$$

For this system we can make use of formulas (16) through (24) of the preceding section and formulas (5) and (29), Sec. 6, with the difference that the total mass of the system $m_1 + m_0$[1] should be substituted everywhere for the mass m.

[1] The basis for this is discussed in Chapter 5.

Substituting F_a from formula (1) into (20), Sec. 7, we may write the following expression for the amplitude of forced vibrations

$$x_a = \frac{m_0 r \omega^2}{(m_1 + m_0)\sqrt{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}} \tag{3}$$

Having determined the extreme value of (3), we see that the maximum amplitude

$$x_{a\,max} = \frac{m_0 r \omega_0^2}{2(m_1 + m_0)\, h \sqrt{\omega_0^2 - h^2}} \tag{4}$$

is attained at the resonant frequency

$$\omega_m = \frac{\omega_0^2}{\sqrt{\omega_0^2 - 2h^2}} > \omega_0 \tag{5}$$

Figure 18*b* shows the displacement response curve plotted according to formula (3). Table 5 illustrates the characteristic features of forced vibrations with centrifugal excitation.

TABLE 5

Parameters		Method of vibration excitation: Excited by a force. Amplitude of exciting force independent of frequency, $F_a = \text{const}$	Method of vibration excitation: Centrifugally excited, $F_a = m_0 r \omega^2$
Amplitude of forced vibrations	general expression	$x_a = \frac{F_a}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}}$	$x_a = \frac{m_0 r \omega^2}{(m_1 + m_0)\sqrt{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}}$
	at $\omega = 0$	$x_{a0} = \frac{F_a}{m\omega_0^2}$	$x_{a0} = 0$
	at $\omega = \infty$	$x_{a\infty} = 0$	$x_{a\infty} = \frac{m_0 r}{m_1 + m_0}$
	at resonance	$x_{a\,max} = \frac{F_a}{2mh\sqrt{\omega_0^2 - h^2}}$	$x_{a\,max} = \frac{m_0 r \omega_0^2}{2(m_1 + m_0)\, h \sqrt{\omega_0^2 - h^2}}$
Displacement resonant frequency		$\omega_m = \sqrt{\omega_0^2 - 2h^2} < \omega_0$	$\omega_m = \frac{\omega_0^2}{\sqrt{\omega_0^2 - 2h^2}} > \omega_0$

Let us now turn to the discussion of kinematically excited forced vibrations. Consider a body *1* (Fig. 19) of mass m connected to fixed stand *4* by spring *2* and damper *3* and free to move over guides *5*. The other side of body *1* is connected by spring *6* and damper *7* with carrier *8* which moves over guides *9*. Both springs and dampers are linear and the guides ideal. The stiffnesses of springs *2* and *6* are c_1 and c_2, respectively, and the coefficients of resistance of dampers *3* and *7* are b_1 and b_2, respectively. A sinusoidal motion is imparted to the carrier by an external factor. Let us denote by z the carrier coordinate and by x the coordinate of body *1* measured from their mean position. The coordinates describe the absolute motions of the carrier and body *1*.

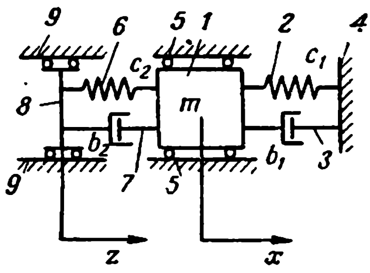


Figure 19

The differential equation of motion takes the form

$$m\ddot{x}+b_1\dot{x}+b_2(\dot{x}-\dot{z})+c_1x+c_2(x-z)=0 \tag{6}$$

According to what has been said above

$$z=z_a\cos\omega t \tag{7}$$

and we can rewrite Eq. (7) in the form

$$m\ddot{x}+(b_1+b_2)\dot{x}+(c_1+c_2)x=z_a(c_2\cos\omega t-b_2\omega\sin\omega t) \tag{8}$$

or

$$\ddot{x}+2h\dot{x}+\omega_0^2x=P\cos(\omega t+\psi) \tag{9}$$

where

$$h=\frac{b_1+b_2}{2m}; \qquad \omega_0=\sqrt{\frac{c_1+c_2}{m}} \tag{10}$$

$$P=\frac{z_a}{m}\sqrt{c_2^2+b_2^2\omega^2}; \quad \psi=\tan^{-1}\frac{b_2\omega}{c_2} \tag{11}$$

We can consider the motion of body *1* with respect to the carrier instead of its absolute motion. In this case the coordinate of the relative motion is

$$y=x-z \tag{12}$$

Substituting (12) into Eq. (6) and using (7), we obtain

$$\ddot{y}+2h\dot{y}+\omega_0^2y=Q\cos(\omega t+\chi) \tag{13}$$

where h and ω_0 are determined by the relations (10), and

$$\left.\begin{aligned}Q&=\frac{z_a}{m}\sqrt{(c_1-m\omega^2)^2+b_1^2\omega^2}\\ \chi&=\tan^{-1}\frac{b_1\omega}{c_1-m\omega^2}\end{aligned}\right\} \tag{14}$$

Thus, the differential equations of kinematically excited motions, whether absolute (9) or relative (13), are identical in their final form with the equation of a motion excited by an external force [see Eq. (17), Sec. 7], except for the initial phases of the exciting factors which in the case of kinematic excitation are, generally speaking, different from zero.

The frequency dependences of the amplitudes P and Q or phases ψ and χ of the exciting factors make it possible to obtain various displacement and phase response characteristics[1].

The amplitude of vibration of the mass of the system (Fig. 20) with positive motion is determined by the kinematics of the drive

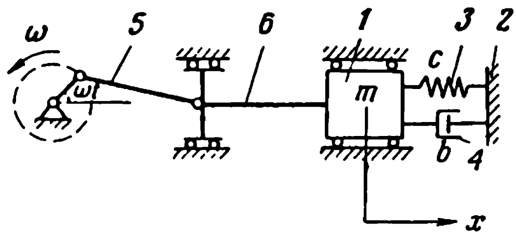


Figure 20

and is independent of frequency. Body *1* of mass m is connected to fixed stand *2* by spring *3* and damper *4*. Sinusoidally varying vibrations of body *1* are produced by the crank gear with connecting rod *5* and piston rod *6*. Assuming that the ratio of the crank radius to the connecting-rod length $r/l \ll 1$, we can write the equation of motion of body *1* as follows:

$$x = r\cos\omega t \qquad (15)$$

where ω is the constant angular velocity of crank rotation.

Using D'Alembert's principle, we can write the condition of dynamic equilibrium of body *1*:

$$J + B + S + F = 0 \qquad (16)$$

where $J = -m\ddot{x}$ = inertia force

$B = -b\dot{x}$ = damper force

$S = -cx$ = spring force

$F =$ force applied to piston rod *6*.

Let the spring force in the mean position of body *1* be zero; substituting (15) into Eq. (16), we now obtain

$$F = F_a\cos(\omega t + \varphi) \qquad (17)$$

[1] See graphs in Section 13.

where

$$\left.\begin{aligned} F_a &= mr\sqrt{(\omega_0^2-\omega^2)^2+4h^2\omega^2} \\ \varphi &= \tan^{-1}\frac{2h\omega}{\omega_0^2-\omega^2} \end{aligned}\right\} \quad (18)$$

The second of formulas (18) is identical with formula (21), Sec. 7; ω_0 and h are defined by formulas (5) and (29), Sec. 6.

The amplitude of the force F developed by the drive is minimum at $\omega = \sqrt{\omega_0^2-2h^2}$ and this minimum value is

$$F_{a\,min} = 2mrh\sqrt{\omega_0^2-h^2} \quad (19)$$

In the absence of dissipative resistances $F_{a\,min}=0$ at $\omega=\omega_0$.

When $\omega=0$

$$F_{a0} = mr\omega_0^2 \quad (20)$$

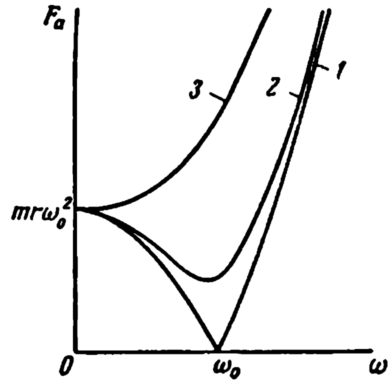


Figure 21

It can be seen that the resonance in the system in question reveals itself by diminishing the force in the drive. If the tuning is near to the resonance condition, the drive force may prove considerably larger at starting than the necessary steady-state value of the force amplitude.

The amplitude response curves of the drive force are shown in Fig. 21. Curve *1* corresponds to $h=0$, curve *2* to $h<\omega_0/\sqrt{2}$ and curve *3* to $h>\omega_0/\sqrt{2}$.

9. Polyharmonic and Nonperiodic Excitations

Sinusoidal exciting actions form an important, though a special and very narrow, class of excitations. Nonsinusoidal periodic or nonperiodic excitations are often encountered in practice. We shall now discuss three methods of integrating differential equations of nonsinusoidally excited motions: expansion of the exciting force in a Fourier series; the method of fitting (piecewise analytical representation and "sewing" or "glueing" of the pieces); and Lagrange's method (variation of integration constants).

Let the system (see Fig. 10) be subjected to the action of the force $F(t)$ applied to body *1*. The force $F(t)$ is a known function of time. The motion of the system is described by the differential equation

$$\ddot{x}+2h\dot{x}+\omega_0^2x=\frac{1}{m}F(t) \quad (1)$$

If the exciting force is periodic, with the period $2\pi/\omega$, then the force can be expanded in a Fourier series[1]:

$$F(t) = \frac{1}{2} F_0 + \sum_{n=1}^{\infty} F_n \cos(n\omega t - \psi_n) \tag{2}$$

where F_0, F_n, ψ_n are obtained from formulas (4) and (2), Sec. 4.

Using the principle of superposition, we can write down the general solution of Eq. (1):

$$x = x_1 + x_2 + \frac{F_0}{2m\omega_0^2} + \frac{1}{m} \sum_{n=1}^{\infty} \frac{F_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4h^2n^2\omega^2}} \cos(n\omega t - \psi_n - \varphi_n) \tag{3}$$

where

$$\varphi_n = \tan^{-1} \frac{2hn\omega}{\omega_0^2 - n^2\omega^2} \tag{4}$$

and x_1 and x_2 are two linearly independent integrals, representing the natural vibrations of the system, for example, when $h < \omega_0$ the sum $x_1 + x_2$ is expressed by the first term on the right-hand side of expression (18), Sec. 7.

If the Fourier series representing the force $F(t)$ is infinite, then the series representing the forced vibrations is also infinite. To obtain a solution in finite form, the latter series must be summated, this being in many cases a difficult or even impossible task. This may sometimes be an essential defect of the method. It should be noted that under the action of a polyharmonic force a system with one degree of freedom can have a large number of resonances corresponding to the number of harmonic terms in the expansion of $F(t)$.

Consider, for instance, the periodic force varying according to the law:

$$\left.\begin{aligned} F(t) &= \Phi \quad \text{at} \quad \frac{2\nu\pi}{\omega} \leqslant t \leqslant \frac{(2\nu+1)\pi}{\omega} \\ F(t) &= -\Phi \quad \text{at} \quad \frac{(2\nu+1)\pi}{\omega} \leqslant t \leqslant \frac{(2\nu+2)\pi}{\omega} \end{aligned}\right\} \quad (\nu = 0, 1, 2, \ldots) \tag{5}$$

The graph representing $F(t)$ is shown in Fig. 22*a*. We now expand this force in a Fourier series:

$$F = \frac{4\Phi}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\omega t}{2n-1} \tag{6}$$

---

[1] The method of expansion in a Fourier series is also applicable when the force is nonperiodic and defined in a finite time interval. The solution thus obtained is valid only within this interval.

From this we obtain the following particular integral of Eq. (1) corresponding to forced vibrations:

$$x=\frac{4\Phi}{\pi m}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\omega t}{(2n-1)\sqrt{[\omega_0^2-(2n-1)^2\omega^2]^2+4(2n-1)^2h^2\omega^2}} \tag{7}$$

In a special case, when $h=0$,

$$x=\frac{4\Phi}{\pi m}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\omega t}{(2n-1)[\omega_0^2-(2n-1)^2\omega^2]} \tag{8}$$

The use of the method of fitting (see Sec. 20) is advisable in cases where the function $F(t)$ can be broken down into a number of parts from 0 to t_1, from t_1 to t_2, etc., so that the integral of Eq. (1) can be obtained for each of the time intervals.

We shall now use this method to determine the stationary vibrations of the conservative system (see Fig. 7) excited by the force $F(t)$ which is defined by the conditions (5). The differential equation of motion (1) for one period of action of the exciting force takes the form

$$\ddot{x}+\omega_0^2x=\begin{cases}\dfrac{\Phi}{m} \text{ at } 0\leqslant t\leqslant\dfrac{\pi}{\omega}\\[2mm] -\dfrac{\Phi}{m} \text{ at } \dfrac{\pi}{\omega}\leqslant t\leqslant\dfrac{2\pi}{\omega}\end{cases} \tag{9}$$

The general solution for the time interval $0\leqslant t\leqslant\pi/\omega$ has the form

$$x=C_1\cos\omega_0t+C_2\sin\omega_0t+\frac{\Phi}{m\omega_0^2} \tag{10}$$

whence

$$\dot{x}=-C_1\omega_0\sin\omega_0t+C_2\omega_0\cos\omega_0t \tag{11}$$

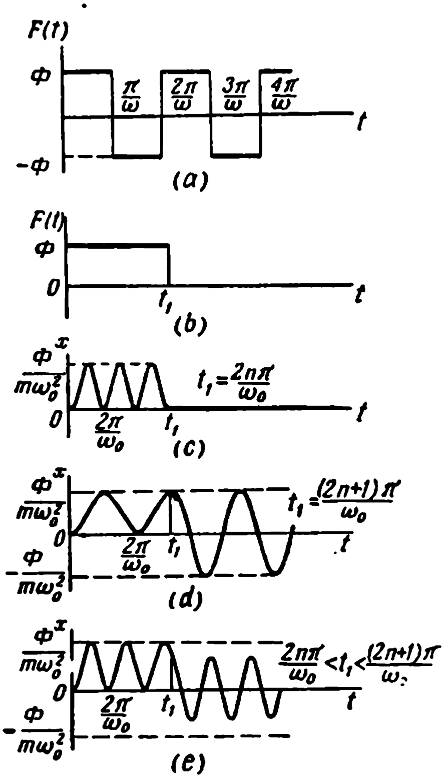


Figure 22

Suppose that for steady-state forced vibrations existing before the moment $t=0$ the initial conditions at the moment $t=0$ are $x=x_0$ and $\dot{x}=\dot{x}_0$ whose values are to be determined in the course of solving.

With account taken of the initial conditions the expressions (10) and (11) become

$$\left.\begin{aligned} x &= \left(x_0 - \frac{\Phi}{m\omega_0^2}\right)\cos\omega_0 t + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t + \frac{\Phi}{m\omega_0^2} \\ \dot{x} &= -\left(x_0 - \frac{\Phi}{m\omega_0^2}\right)\omega_0 \sin\omega_0 t + \dot{x}_0\cos\omega_0 t \end{aligned}\right\} \tag{12}$$

Further we have to write the general solution for the interval $\pi/\omega \leqslant t \leqslant 2\pi/\omega$ and equate the values of x and $\dot{x}$ at the beginning of the interval to the values of x and $\dot{x}$ at the end of the preceding interval.

Note that in order to save time we can make use of the fact that the function $F(t)$ over the half-period $\pi/\omega \leqslant t \leqslant 2\pi/\omega$ differs only in sign from its value over the half-period $0 \leqslant t \leqslant \pi/\omega$. Owing to symmetry with steady-state forced vibrations the initial conditions at the beginning of each of the half-periods differ only in sign, i.e., at $t = \pi/\omega$ they are $x = -x_0$ and $\dot{x} = -\dot{x}_0$[1]. At the same time the initial conditions for the second half-period $\pi/\omega \leqslant t \leqslant 2\pi/\omega$ are the conditions at the end of the first half-period $0 \leqslant t \leqslant \pi/\omega$. Introducing these conditions into (12), we obtain a system of equations

$$\left(1+\cos\frac{\pi\omega_0}{\omega}\right)x_0 + \left(\frac{1}{\omega_0}\sin\frac{\pi\omega_0}{\omega}\right)\dot{x}_0 = -\frac{\Phi}{m\omega_0^2}\left(1-\cos\frac{\pi\omega_0}{\omega}\right)$$

$$\left(\sin\frac{\pi\omega_0}{\omega}\right)x_0 - \frac{1}{\omega_0}\left(1+\cos\frac{\pi\omega_0}{\omega}\right)\dot{x}_0 = \frac{\Phi}{m\omega_0^2}\sin\frac{\pi\omega_0}{\omega}$$

Hence

$$\left.\begin{aligned} x_0 &= 0 \\ \dot{x}_0 &= -\frac{\Phi}{m\omega_0}\tan\frac{\pi\omega_0}{2\omega} \end{aligned}\right\} \tag{13}$$

Substituting (13) into the first of expressions (12) and having in mind the symmetry pointed out above, we obtain the result required, corresponding to steady-state forced vibrations:

$$\left.\begin{aligned} x &= \frac{\Phi}{m\omega_0^2}\left[1 - \sec\frac{\pi\omega_0}{2\omega}\cos\left(\omega t - \frac{\pi\omega_0}{2\omega}\right)\right] \\ &\quad \text{at } \frac{2\nu\pi}{\omega} \leqslant t \leqslant \frac{(2\nu+1)\pi}{\omega} \\ x &= -\frac{\Phi}{m\omega_0^2}\left[1 - \sec\frac{\pi\omega_0}{2\omega}\cos\left(\omega t - \frac{\pi\omega_0}{2\omega}\right)\right] \\ &\quad \text{at } \frac{(2\nu+1)\pi}{\omega} \leqslant t \leqslant \frac{(2\nu+2)\pi}{\omega} \end{aligned}\right\} \tag{14}$$

---

[1] The conditions of periodicity consisting in that the x and $\dot{x}$ values at the beginning and at the end of the period are equal are here automatically satisfied.

The relations (14) are equivalent to (8), which can be ascertained by expanding (14) into Fourier series. They are useful as the solution is presented in finite form. The two forms of the solution are applicable only to steady-state forced vibrations. They are not valid at resonance when

$$\frac{\omega_0}{\omega} = 2n+1, \quad (n=0,\ 1,\ 2)$$

since in a conservative system the vibrations increase continuously at resonance. Formula (8) shows that only the vibrations of the resonant harmonic are increased.

The method of variation of parameters is universally applicable. It reduces the problem of solving inhomogeneous linear equations to quadratures[1].

Let $x_1(t)$ and $x_2(t)$ be two linearly independent integrals of the homogeneous equation (28), Sec. 6. Its general solution can be written as follows:

$$x = C_1 x_1(t) + C_2 x_2(t) \tag{15}$$

Using Lagrange's method, the general solution of the inhomogeneous equation (1) is taken in the same form, but C_1 and C_2 are not considered as constants; they are now functions of the argument (time) which are to be determined, i.e., $C_1 = u_1(t)$, $C_2 = u_2(t)$. Thus

$$x = u_1(t)\, x_1(t) + u_2(t)\, x_2(t) \tag{16}$$

Since there are two functions to be sought, we can introduce an arbitrary supplementary condition in the form of the equation

$$\dot{u}_1(t)\, x_1(t) + \dot{u}_2(t)\, x_2(t) = 0 \tag{17}$$

Substituting expression (16) and its first and second derivatives into differential equation (1) and taking account of condition (17), we obtain

$$u_1(t)\,[\ddot{x}_1(t) + 2h\dot{x}_1(t) + \omega_0^2 x_1(t)] + u_2(t)\,[\ddot{x}_2(t) + 2h\dot{x}_2(t) +$$
$$+ \omega_0^2 x_2(t)] + \dot{u}_1(t)\,\dot{x}_1(t) + \dot{u}_2(t)\,\dot{x}_2(t) = \frac{1}{m} F(t)$$

The expressions in brackets are equal to zero as $x_1(t)$ and $x_2(t)$ are integrals of the corresponding homogeneous equation. Accordingly

$$\dot{u}_1(t)\,\dot{x}_1(t) + \dot{u}_2(t)\,\dot{x}_2(t) = \frac{1}{m} F(t) \tag{18}$$

[1] The method is suitable for integrating linear differential equations of any order, including those with variable coefficients.

From the simultaneous equations (17) and (18) we obtain $\dot{u}_1(t)$ and $\dot{u}_2(t)$ and integrating them we find the functions required:

$$\left.\begin{aligned} u_1(t) &= -\frac{1}{m}\int_{t_0}^{t}\frac{F(z)\,x_2(z)}{x_1(z)\,\dot{x}_2(z)-\dot{x}_1(z)\,x_2(z)}\,dz + C_1' \\ u_2(t) &= \frac{1}{m}\int_{t_0}^{t}\frac{F(z)\,x_1(z)}{x_1(z)\,\dot{x}_2(z)-\dot{x}_1(z)\,x_2(z)}\,dz + C_2' \end{aligned}\right\} \tag{19}$$

where C_1' and C_2' are integration constants whose values are determined from the initial conditions.

The denominators of the integrands cannot be zero as $x_1(t)$ and $x_2(t)$ are linearly independent. Consequently, the simultaneous equations (17) and (18) always have definite solutions. The lower integration limit t_0 is the moment of time at which the force $F(t)$ was applied to the system and the upper limit t is the running moment.

Using formulas (19), we can write the required expression of the general solution (16):

$$x = C_1'x_1(t) + C_2'x_2(t) + \frac{1}{m}\int_{t_0}^{t}\frac{F(z)\,[x_1(z)\,x_2(t)-x_2(z)\,x_1(t)]}{x_1(z)\,\dot{x}_2(z)-x_2(z)\,\dot{x}_1(z)}\,dz \tag{20}$$

If in Eq. (1) $h=0$, we may assume $x_1(t)=\cos\omega_0 t$, $x_2(t)=\sin\omega_0 t$. In this case for the initial conditions (7), Sec. 6, at $t_0=0$ the solution (20) takes the form

$$x = x_0\cos\omega_0 t + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t + \frac{1}{m\omega_0}\int_0^t F(z)\sin\omega_0(t-z)\,dz \tag{21}$$

For example, if the system (see Fig. 22*b*) with zero initial conditions is acted upon at the moment $t=0$ by the force

$$F(t)=\begin{cases}\Phi & \text{at}\quad 0\leqslant t\leqslant t_1\\ 0 & \text{at}\quad t>t_1\end{cases}$$

then, using relation (21), we obtain the following solution for the time interval $0\leqslant t\leqslant t_1$:

$$x = \frac{\Phi}{m\omega_0}\int_0^t \sin\omega_0(t-z)\,dz = \frac{\Phi}{m\omega_0^2}(1-\cos\omega_0 t)$$

whence

$$\dot{x} = \frac{\Phi}{m\omega_0}\sin\omega_0 t$$

For the time $t>t_1$ we have

$$x = x_1\cos\omega_0(t-t_1) + \frac{\dot{x}_1}{\omega_0}\sin\omega_0(t-t_1)$$

where

$$x_1=\frac{\Phi}{m\omega_0^2}(1-\cos\omega_0 t_1);\quad \dot{x}_1=\frac{\Phi}{m\omega_0}\sin\omega_0 t_1$$

During the time interval $0 \leqslant t \leqslant t_1$ the system vibrates about a new equilibrium position displaced by the amount of static deformation under the action of the force Φ,

$$x_{st}=\frac{\Phi}{m\omega_0^2}=\frac{\Phi}{c}$$

with the amplitude x_{st} so that $x_{max} = 2x_{st}$ and $x_{min} = 0$. The motion of the system at $t > t_1$ depends on the ratio of t_1 to the period of natural vibrations of the system. If t_1 is a whole number of periods of natural vibration (an even number of half-periods π/ω_0), i.e., $t_1 = 2n\pi/\omega_0$ ($n = 1, 2, \ldots$), then at $t > t_1$ the system remains at rest as $x_1 = \dot{x}_1 = 0$ (see Fig. 22*c*). If t_1 is equal to an odd number of half-periods of natural vibration, i.e., $t_1=\frac{(2n-1)\pi}{\omega_0}$, then at $t > t_1$ the system vibrates at the maximum amplitude $x_a = 2x_{st}$ as shown in Fig. 22*d*. If the time t_1 has an intermediate value $\frac{(2n-1)\pi}{\omega_0} < t_1 < \frac{2n\pi}{\omega_0}$, then at $t > t_1$ the system vibrates with the amplitude $0 < x_a < 2x_{st}$ (Fig. 22*e*).

10. General Approach to Setting up of Differential Equations of Motion

The motion of systems having one degree of freedom has been discussed on concrete examples. We now select a more general approach for the discussion of systems with several degrees of freedom (two or more, but the number being finite). Consider a system containing N point masses. Their positions are determined by $3N$ cartesian coordinates of the points in an inertial system of coordinates: $x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N, z_1, z_2, \ldots, z_N$. Let there be s independent constraints imposed on the system and limiting its motion. The number of degrees of freedom will be

$$n=3N-s \tag{1}$$

If the constraints are stationary, as will be assumed in the following discussion, they are determined by the equations

$$f_p(x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N, z_1, z_2, \ldots, z_N)=0 \tag{2}$$
$$(p=1, 2, \ldots, s)$$

Any n cartesian coordinates can be taken as independent generalized coordinates of the system. They will specify completely its

position as the remaining s coordinates are determined from Eq. (2) using these n coordinates. However, any other q_i coordinates ($i = 1, 2, \ldots, n$) may be taken as generalized coordinates related to the cartesian coordinates as follows

$$q_i = q_i(x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N, z_1, z_2, \ldots, z_N) \quad (i=1, 2, \ldots, n) \tag{3}$$

We use here only the stationary transformation of coordinates, the time being absent in explicit form on the right-hand sides of Eq. (3).

The generalized coordinates q_i must determine the position of the system completely. Therefore all the cartesian coordinates can be determined from the n relations (3) and the s constraint equations (2)

$$\left.\begin{aligned} x_k &= x_k(q_1, q_2, \ldots, q_n) \\ y_k &= y_k(q_1, q_2, \ldots, q_n) \\ z_k = z_k(q_1, q_2, \ldots, q_k), &\quad (k=1, 2, \ldots, N) \end{aligned}\right\} \tag{4}$$

The motion of the system can be described by Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i, \ (i=1,\ 2, \ldots, n) \tag{5}$$

where L = Lagrange's function equal to the excess of the kinetic energy T of the system over its potential energy Π, i.e.,

$$L = T - \Pi \tag{6}$$

Q_i = generalized forces determined by the formulas

$$Q_i = \sum_{k=1}^{N}\left(X_k\frac{\partial x_k}{\partial q_i} + Y_k\frac{\partial y_k}{\partial q_i} + Z_k\frac{\partial z_k}{\partial q_i}\right) \quad (i=1, 2, \ldots, n) \tag{7}$$

The quantities X_k, Y_k, Z_k on the right-hand side of Eq. (7) are projections of the resultant of the nonpotential forces applied to the kth point mass of the system. In the case of the free motion of a conservative system, the first to be considered, $Q_i = 0$.

The kinetic energy of the system can be determined by the formula

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k(\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2) \tag{8}$$

where m_k is the kth point mass.

Differentiating expressions (4) yields

$$\dot{x}_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i} \dot{q}_i; \quad \dot{y}_k = \sum_{i=1}^{n} \frac{\partial y_k}{\partial q_i} \dot{q}_i$$

$$\dot{z}_k = \sum_{i=1}^{n} \frac{\partial z_k}{\partial q_i} \dot{q}_i \tag{9}$$

Substituting expressions (9) into formula (8), we have

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \dot{q}_i \dot{q}_j \tag{10}$$

where $a_{ij} = a_{ji}$ are coefficients depending only on the generalized coordinates (in particular cases a_{ij} and a_{ji} are constants).

It follows from formula (8) that the kinetic energy is always a positive quantity and vanishes only when the velocities of all the point masses of the system are zero. In this case all the generalized velocities become zero, as can be seen on differentiating relations (3) with respect to time. Thus, the quadratic form on the right-hand side of expression (10) is positively definite. Consequently, its coefficients a_{ij} satisfy Silvester's criteria which require that the determinant and all the diagonal corner minors of the coefficient matrix

$$a = \left\| \begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{matrix} \right\| \tag{11}$$

be positive, i.e.,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots, \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} > 0 \tag{12}$$

Free vibrations are possible only about the position of stable equilibrium. There are three kinds of equilibrium: *stable*, *unstable* and *indifferent*. Examples of stable equilibrium are shown in Fig. 23*a* and *d* (a heavy ball in a hollow and a pendulum in its lower position); those of unstable equilibrium are depicted in Fig. 23*b* and *e* (a ball on the top of a hillock and a pendulum in its upper position), and an example of indifferent equilibrium is shown in Fig. 23*c* (a ball on a horizontal plane).

A system displaced from its position of stable equilibrium, generally speaking, not too far away, returns to it. A system displaced

from its position of unstable equilibrium deviates from it still further. A system slightly displaced from its position of indifferent equilibrium remains in the new position.

The potential energy of a system can be represented by a function of the generalized coordinates:

$$\Pi = \Pi (q_1, q_2, \ldots, q_n) \tag{13}$$

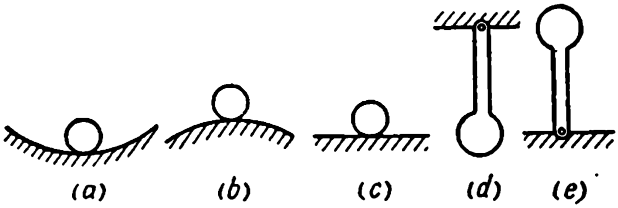


Figure 23

The potential energy of a conservative system in the position of equilibrium has its extreme value, i.e.,

$$\left(\frac{\partial \Pi}{\partial q_1}\right)_{q_i=q_i^{(0)}}=0;\ \left(\frac{\partial \Pi}{\partial q_2}\right)_{q_i=q_i^{(0)}}=0;\ \ldots \quad \left(\frac{\partial \Pi}{\partial q_n}\right)_{q_i=q_i^{(0)}}=0 \tag{14}$$

where $q_i^{(0)}$ are the generalized coordinates at the equilibrium position.

According to the Lagrange-Dirichlet theorem it is the minimum of potential energy that corresponds to the position of stable equilibrium of a conservative system.

Taking the stable equilibrium position as the origin of the generalized coordinates, i.e., $q_i^{(0)} = 0$, $(i = 1, 2, \ldots, n)$, let us expand the potential energy in the neighbourhood of this point in a Taylor's series:

$$\Pi = \Pi^{(0)} + \sum_{i=1}^{n} k_i q_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j + \ldots \tag{15}$$

The potential energy is determined accurate to a constant term. Equality (5) shows that this additive constant plays no part in the setting up of the equations of motion. We assume, accordingly, the value of the potential energy in the stable equilibrium position of the system to be zero, i.e., $\Pi^{(0)} = 0$. The coefficients k_i of the first powers of the generalized coordinates are also zero, as

$$k_i = \left(\frac{\partial \Pi}{\partial q_i}\right)_{q_i=0}$$

Potential energy expansions of linear systems do not contain terms with the powers of the generalized velocities higher than the

second. In considering small[1] vibrations of nonlinear systems about the position of stable equilibrium the terms containing powers higher than the second may often be neglected as small quantities of higher orders if not all of the coefficients k_{ij} of the quadratic terms are zero. Thus the potential energy is expressed by the quadratic form

$$\Pi = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j, \quad (k_{ij} = k_{ji}) \tag{16}$$

this form being positively definite, which follows from the Lagrange-Dirichlet theorem. Consequently, the coefficients k_{ij} satisfy Silvester's criteria, i.e.,

$$k_{11} > 0, \quad \begin{vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{vmatrix} > 0, \ldots, \quad \begin{vmatrix} k_{11} \ldots k_{1n} \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ k_{n1} \ldots k_{nn} \end{vmatrix} > 0 \tag{17}$$

As has been pointed out above, the coefficients a_{ij} of the quadratic forms of generalized velocities in the expression for kinetic energy (10) depend in general on the generalized coordinates. Expanding them in a Taylor's series in the neighbourhood of the point of stable equilibrium, we obtain

$$a_{ij} = a_{ij}^{(0)} + \sum_{p=1}^{n} a_{ij}^{(1)} q_p + \ldots \tag{18}$$

Only the constant term $a_{ij}^{(0)}$ is present in this series in the case of linear systems. For small vibrations of nonlinear systems the terms containing the power 1 and higher of the generalized coordinates may often be neglected as small quantities of higher order. In what follows it is assumed that a_{ij} = const.

Substituting into expression (6) the values of the kinetic and potential energies from (10) and (16) and differentiating, as indicated by Eq. (5), we obtain the following differential equations of motion:

$$\sum_{j=1}^{n} a_{ij} \ddot{q}_j + \sum_{j=1}^{n} k_{ij} q_j = Q_i, \quad (i = 1, 2, \ldots, n) \tag{19}$$

The dissipative forces can be taken into account in calculating the generalized forces from formulas (7) and then one can use Eq. (5).

An alternative method of setting up the equations of motion is applicable to linear systems. The dissipative function is constructed

[1] This refers to quantities small compared with unity. This treatment has a meaning if the small quantity is dimensionless. Thus, 1 mm is a small quantity, 10^{-3}, if the unit of measurement is the metre, but 1 mm is a large quantity, 10^3, if the micron is used as the unit of measurement.

as a positively definite quadratic form of generalized velocities

$$\Phi=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\dot{q}_i\dot{q}_j, \qquad (b_{ij}=b_{ji}) \tag{20}$$

and its partial derivative with respect to any of the generalized velocities, taken with an opposite sign, is the dissipative force B corresponding to this generalized velocity:

$$B_i=-\frac{\partial\Phi}{\partial\dot{q}_i}, \qquad (i=1, 2, \ldots, n) \tag{21}$$

In this case Eq. (5) may be replaced by the differential equations of the form

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}-\frac{\partial L}{\partial q_i}+\frac{\partial\Phi}{\partial\dot{q}_i}=Q_i, \qquad (i=1, 2, \ldots, n) \tag{22}$$

where the generalized forces Q_i depend only on time.

Since the dissipative function is positively definite, the coefficients b_{ij} satisfy Silvester's criteria, i.e.,

$$b_{11}>0, \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}>0, \ldots, \begin{vmatrix} b_{11} & \ldots & b_{1n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ b_{n1} & \ldots & b_{nn} \end{vmatrix}>0 \tag{23}$$

Differentiating, as indicated by the left-hand side of expressions (22), we obtain the following system of differential equations:

$$\sum_{j=1}^{n} a_{ij}\ddot{q}_j+\sum_{j=1}^{n} b_{ij}\dot{q}_j+\sum_{j=1}^{n} k_{ij}q_j=Q_i \tag{24}$$

$$(i=1, 2, \ldots, n)$$

11. Systems with Two and More Degrees of Freedom

For free vibrations of a conservative system the right-hand sides of Eqs. (19), Sec. 10, are equal to zero: $Q_i=0$ ($i=1, 2, \ldots, n$). The motion is then described by n simultaneous ordinary linear differential equations of the second order:

$$\sum_{j=1}^{n} a_{ij}\ddot{q}_j+\sum_{j=1}^{n} k_{ij}q_j=0 \tag{1}$$

$$(i=1, 2, \ldots, n;\ a_{ij}=a_{ji};\ k_{ij}=k_{ji})$$

or in extended form

$$\left.\begin{array}{l}
a_{11}\ddot{q}_1+a_{12}\ddot{q}_2+\ldots+a_{1n}\ddot{q}_n+k_{11}q_1+k_{12}q_2+\ldots+k_{1n}q_n=0\\
a_{21}\ddot{q}_1+a_{22}\ddot{q}_2+\ldots+a_{2n}\ddot{q}_n+k_{21}q_1+k_{22}q_2+\ldots+k_{2n}q_n=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
a_{n1}\ddot{q}_1+a_{n2}\ddot{q}_2+\ldots+a_{nn}\ddot{q}_n+k_{n1}q_1+k_{n2}q_2+\ldots+k_{nn}q_n=0
\end{array}\right\} \tag{2}$$

The solutions of these simultaneous differential equations are

$$q_i=A_i\cos(\Omega t-\varphi), \qquad (i=1, 2, \ldots, n) \tag{3}$$

Substituting into differential equations (2) the expressions for generalized coordinates (3) and their second derivatives, we obtain n simultaneous algebraic homogeneous linear equations in coefficients A_i:

$$\left.\begin{array}{l}
(k_{11}-a_{11}\Omega^2)A_1+(k_{12}-a_{12}\Omega^2)A_2+\ldots+(k_{1n}-a_{1n}\Omega^2)A_n=0\\
(k_{21}-a_{21}\Omega^2)A_1+(k_{22}-a_{22}\Omega^2)A_2+\ldots+(k_{2n}-a_{2n}\Omega^2)A_n=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
(k_{n1}-a_{n1}\Omega^2)A_1+(k_{n2}-a_{n2}\Omega^2)A_2+\ldots+(k_{nn}-a_{nn}\Omega^2)A_n=0
\end{array}\right\} \tag{4}$$

It is well known that the necessary and sufficient condition for the existence of non-zero solutions of such simultaneous equations is that its determinant be equal to zero:

$$\begin{vmatrix}
k_{11}-a_{11}\Omega^2 & k_{12}-a_{12}\Omega^2 & & k_{1n}-a_{1n}\Omega^2\\
k_{21}-a_{21}\Omega^2 & k_{22}-a_{22}\Omega^2 & \ldots & k_{2n}-a_{2n}\Omega^2\\
\cdots & \cdots & & \cdots\\
k_{n1}-a_{n1}\Omega^2 & k_{n2}-a_{n2}\Omega^2 & \ldots & k_{nn}-a_{nn}\Omega^2
\end{vmatrix}=0 \tag{5}$$

Expression (5) is an equation of nth degree in the square of frequency, Ω^2. Solving it, we obtain n squares of the natural frequencies ($\Omega_1, \Omega_2, \ldots, \Omega_n$) of the mechanical system considered. Expression (5) is termed the *characteristic equation*[1]. In our case all the roots Ω_i^2 of the characteristic equation are real and non-negative. To each frequency square value there correspond two values of the frequency: $\Omega_i=\pm\sqrt{\Omega_i^2}$. We shall use only the positive values because the negative frequencies give us no additional information about the vibrational motion; they are equivalent to the positive ones if the initial phases have been properly selected[2].

[1] The terms *century equation*, *frequency equation* and *secular equation* are also used.

[2] It was shown in Sec. 2 that the sign of angular frequency (angular velocity) must be taken into account in considering periodic rotational motion.

Consider the case where all natural frequencies are different: $\Omega_1 < \Omega_2 < \Omega_3 < \ldots < \Omega_n$. It follows from what has been said that the general solutions of the simultaneous equations (2) can be presented in the form

$$\left.\begin{aligned} q_1 &= A_{11}\cos(\Omega_1 t - \varphi_1) + A_{12}\cos(\Omega_2 t - \varphi_2) + \ldots + \\ &\quad + A_{1n}\cos(\Omega_n t - \varphi_n) \\ q_2 &= A_{21}\cos(\Omega_1 t - \varphi_1) + A_{22}\cos(\Omega_2 t - \varphi_2) + \ldots + \\ &\quad + A_{2n}\cos(\Omega_n t - \varphi_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ q_n &= A_{n1}\cos(\Omega_1 t - \varphi_1) + A_{n2}\cos(\Omega_2 t - \varphi_2) + \ldots + \\ &\quad + A_{nn}\cos(\Omega_n t - \varphi_n) \end{aligned}\right\} \tag{6}$$

The first subscript i in A_{ij} denotes the order number of the generalized coordinate, and the second subscript j the order number of natural frequency.

The solutions of (6) comprise n^2 quantities A_{ij} and n quantities φ_i, i.e., $n(n+1)$ constants in all. The initial conditions q_{i0} and $\dot{q}_{i0}$ yield only $2n$ equations from which the constants can be found. Consequently $n(n-1)$ additional conditions are required. The conditions are easily determined by considering the structure of the simultaneous equations (4) in which one[1] of the coefficients, say A_1, may be taken arbitrarily. Then all the rest can be determined in terms of A_1. Dividing all the terms on the left-hand side of Eq. (4) by A_1 and using the notation $\alpha_2 = A_2/A_1$, $\alpha_3 = A_3/A_1$, $\ldots$, $\alpha_i = A_i/A_1$, $\ldots$, $\alpha_n = \frac{A_n}{A_1}$, we can now determine these $(n-1)$ quantities α_i $(i = 2, 3, \ldots, n)$. But Ω may assume n values: Ω_1, Ω_2, Ω_3, $\ldots$, Ω_i, $\ldots$, Ω_n. Thus, n values of each of the coefficients α_i are determined, viz. α_{i1}, α_{i2}, $\ldots$, α_{in}. Therefore we obtain $n(n-1)$ values of α_{ij}:

$$\alpha_{ij} = \frac{A_{ij}}{A_{1j}} = f_i(k_{rs} - a_{rs}\Omega_j^2) \tag{7}$$

$$(i = 2, 3, \ldots, n;\ j = 1, 2, \ldots, n)$$

The subscripts r and s indicate that the whole set of the coefficients k and a is meant. The ratios α_{ij} are called *distribution coefficients.*

---

[1] Or more than one if there are equal natural frequencies.

Solution (6) can be now rewritten as follows:

$$\left.\begin{aligned} q_1 &= A_1\cos(\Omega_1 t-\varphi_1)+A_2\cos(\Omega_2 t-\varphi_2)+\ldots+\\ &\quad +A_n\cos(\Omega_n t-\varphi_n)\\ q_2 &= A_1\alpha_{21}\cos(\Omega_1 t-\varphi_1)+A_2\alpha_{22}\cos(\Omega_2 t-\varphi_2)+\\ &\quad +\ldots+A_n\alpha_{2n}\cos(\Omega_n t-\varphi_n)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ q_n &= A_1\alpha_{n1}\cos(\Omega_1 t-\varphi_1)+A_2\alpha_{n2}\cos(\Omega_2 t-\varphi_2)+\\ &\quad +\ldots+A_n\alpha_{nn}\cos(\Omega_n t-\varphi_n) \end{aligned}\right\} \tag{8}$$

Note that the subscripts of the coefficient A refer to the order numbers of natural frequencies, in distinction to those used in expressions (3) and (4) which represent the order numbers of generalized coordinates.

The arbitrary constant A_j and φ_j ($2n$ in all) can now be determined by using the $2n$ initial conditions q_{i0} and $\dot{q}_{i0}$. Solutions (8) enable one to note the following basic features of the free motion of a conservative system about the position of stable equilibrium:

1. The motion corresponding to each of the coordinates is a superposition of n sinusoidal vibrations of (in the general case) different frequencies, i.e., the system has n natural frequencies which are determined by its parameters and are independent of the initial conditions.

2. The vibrations of any generalized coordinate q_i at the jth natural frequency are either in phase (if $\alpha_{ij} > 0$) or in opposite phase (if $\alpha_{ij} < 0$) with respect to the coordinate q_1.

3. If the oscillation amplitude of any coordinate at one of the natural frequencies is known, then the amplitudes of oscillations of all the rest at the same frequency are also known since they are interrelated by the distribution coefficients that are independent of initial conditions.

The initial conditions can be selected so that the amplitudes of vibrations at all frequencies, except one, will be zero.

In this case the system will vibrate at one frequency, i.e., sinusoidally. Such vibrations are termed *normal modes*. A system has n normal modes of vibration[1]. Each normal mode of vibration is characterized by its shape (elastic curve) completely determined by the distribution coefficients corresponding to the given frequency and independent of the initial conditions. These shapes are called *normal elastic curves*.

---

[1] The term *natural vibrations* is sometimes used instead of *normal modes of vibration*. This is misleading because natural vibrations, as shown above, may have a number of frequencies.

Thus, in the general case, natural vibrations prove to be multi-frequency vibrations. The reason is that the generalized coordinates used are coupled. The coupling of the coordinates means that free oscillations of one coordinate cannot exist without oscillations of other coordinates coupled with that one. The determination of the so-called normal (i. e., mutually orthogonal)[1] coordinates the oscillations of which are independent of each other is of interest.

It is easily seen that the coupling of natural vibrations is due to the coupling of differential equations (2). For example, the first of the equations is coupled with the second one by the terms containing the coefficients $a_{12} = a_{21}$ and $k_{12} = k_{21}$. The linear transformation of the generalized coordinates

$$q_i' = g_{i1}q_1 + g_{i2}q_2 + \dots + g_{in}q_n \qquad (9)$$
$$(i = 1, 2, \dots, n)$$

resulting in the elimination of the coefficients a_{ij} and k_{ij} with unequal subscripts ($i \neq j$) yields n independent linear differential equations replacing the simultaneous equations (2):

$$a_i'\ddot{q}_1' + k_i'q_i' = 0, \qquad (i = 1, 2, \dots, n) \qquad (10)$$

There is now no need to use double subscripts: $a_i' = a_{ii}'$, $k_i' = k_{ii}'$. What follows is quite straightforward: in each of the normal coordinates the system behaves as if it were a system having one degree of freedom corresponding to this coordinate.

The transformation (9) reduces the quadratic forms (10) and (16), Sec. 10, which are expressions of the kinetic energy in terms of generalized velocities and of the potential energy in terms of generalized coordinates, to the so-called canonical form. The canonical quadratic form is the sum of the squared variables multiplied by constant factors, but it does not comprise terms containing products of the variables:

$$T = \frac{1}{2}\sum_{i=1}^{n} a_i'(\dot{q}_i')^2; \qquad \Pi = \frac{1}{2}\sum_{i=1}^{n} k_i'(q_i')^2 \qquad (11)$$

The natural frequencies of a system cannot depend on what system of coordinates has been chosen because they are determined by the parameters of the system. Therefore the required normal coordinates depend on time as follows:

$$q_i' = A_i \cos(\Omega_i t - \varphi_i), \qquad (i = 1, 2, \dots, n) \qquad (12)$$

[1] These are sometimes called *principal coordinates.*

Substituting equalities (12) into the simultaneous equations (8) we obtain

$$\left.\begin{aligned} q_1 &= q_1' + q_2' + \ldots + q_n' \\ q_2 &= \alpha_{21}q_1' + \alpha_{22}q_2' + \ldots + \alpha_{2n}q_n' \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ q_n &= \alpha_{n1}q_1' + \alpha_{n2}q_2' + \ldots + \alpha_{nn}q_n' \end{aligned}\right\} \qquad (13)$$

The solution of the simultaneous equations (13) for q_i' gives the required transformations (9). The use of normal coordinates greatly simplifies the study of the motion of a system but their determination involves no less labour than the direct solving of the problem using the initial coordinates q_i.

The above is better illustrated by considering a system having two degrees of freedom. Its kinetic and potential energies are determined by the expressions

$$\begin{aligned} T &= \frac{1}{2}\left(a_1\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_2\dot{q}_2^2\right) \\ \Pi &= \frac{1}{2}\left(k_1q_1^2 + 2k_{12}q_1q_2 + k_2q_2^2\right) \end{aligned} \qquad (14)$$

(since $a_{12} = a_{21}$ and $k_{12} = k_{21}$), Silvester's criteria yielding the relations

$$a_1 > 0;\ a_1a_2 - a_{12}^2 > 0;\ k_1 > 0;\ k_1k_2 - k_{12}^2 > 0 \qquad (15)$$

The differential equations of free vibrations now take the form

$$\left.\begin{aligned} a_1\ddot{q}_1 + a_{12}\ddot{q}_2 + k_1q_1 + k_{12}q_2 &= 0 \\ a_{12}\ddot{q}_1 + a_2\ddot{q}_2 + k_{12}q_1 + k_2q_2 &= 0 \end{aligned}\right\} \qquad (16)$$

Substituting integrals of type (3) into (16), we get

$$\left.\begin{aligned} (k_1 - a_1\Omega^2)A_1 + (k_{12} - a_{12}\Omega^2)A_2 &= 0 \\ (k_{12} - a_{12}\Omega^2)A_1 + (k_2 - a_2\Omega^2)A_2 &= 0 \end{aligned}\right\} \qquad (17)$$

Hence the characteristic equation

$$(k_1 - a_1\Omega^2)(k_2 - a_2\Omega^2) - (k_{12} - a_{12}\Omega^2)^2 = 0 \qquad (18)$$

is a quadratic in Ω^2. Solving it, we obtain two roots, Ω_1^2, and Ω_2^2. Both are real and positive.

Let us write the general integral in the following form:

$$\left.\begin{aligned} q_1 &= B_1\cos(\Omega_1t - \varphi_1) + B_2\cos(\Omega_2t - \varphi_2) \\ q_2 &= B_1\alpha_1\cos(\Omega_1t - \varphi_1) + B_2\alpha_2\cos(\Omega_2t - \varphi_2) \end{aligned}\right\} \qquad (19)$$

where the distribution coefficients

$$\begin{aligned} \alpha_1 &= -\frac{k_1 - a_1\Omega_1^2}{k_{12} - a_{12}\Omega_1^2} = -\frac{k_{12} - a_{12}\Omega_1^2}{k_2 - a_2\Omega_1^2} \\ \alpha_2 &= -\frac{k_1 - a_1\Omega_2^2}{k_{12} - a_{12}\Omega_2^2} = -\frac{k_{12} - a_{12}\Omega_2^2}{k_2 - a_2\Omega_2^2} \end{aligned} \qquad (20)$$

The arbitrary constants B_1, B_2, φ_1, φ_2 can be determined from the initial conditions q_{10}, $\dot{q}_{10}$, q_{20}, $\dot{q}_{20}$.

The coupling of differential equations (16) is due to the terms that contain a_{12} and k_{12}. The expressions

$$\chi_a = \frac{a_{12}}{\sqrt{a_1 a_2}}; \quad \chi_k = \frac{k_{12}}{\sqrt{k_1 k_2}} \tag{21}$$

are called *coupling coefficients*. The first coefficient determines the inertia coupling, and second the positional one. Coefficients of dissipative coupling may appear in dissipative systems. The larger the absolute values of the coupling coefficients, the closer are the generalized coordinates coupled. Generally, the values of the coupling

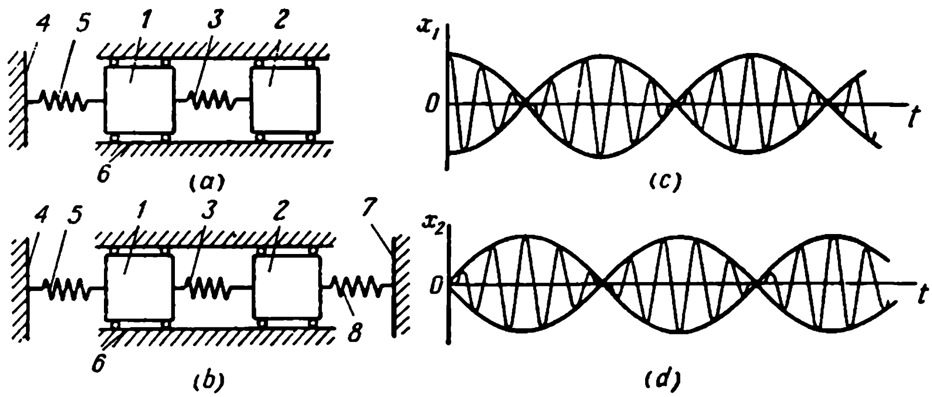


Figure 24

coefficients vary between the limits $-1 < \chi < 1$. There is no coupling when $\chi = 0$.

The normal coordinates (12) are determined by solving the system of equations

$$\left.\begin{aligned} q_1 &= q_1' + q_2' \\ q_2 &= \alpha_1 q_1' + \alpha_2 q_2' \end{aligned}\right\} \tag{22}$$

which is a special case of Eqs. (13).

From (22) we obtain

$$q_1' = -\frac{\alpha_2 q_1 - q_2}{\alpha_1 - \alpha_2}, \quad q_2' = \frac{\alpha_1 q_1 - q_2}{\alpha_1 - \alpha_2} \tag{23}$$

Consider, as an example, the system shown in Fig. 24*a*. Two elements *1* and *2* of masses m_1 and m_2, respectively, are linked by spring *3* whose stiffness coefficient is c_2. Element *1* is also connected to fixed stand *4* by spring *5* with a stiffness coefficient c_1. The ideal guides *6* allow the elements to move only horizontally in the plane of the drawing.

Let the displacements from the position of stable equilibrium x_1 and x_2 (say, to the right) of the elements *1* and *2*, respectively, be selected as general coordinates.

The expressions of the kinetic and potential energies of the system can be written as follows:

$$T=\frac{1}{2}(m_1\dot{x}_1^2+m_2\dot{x}_2^2)$$

$$\Pi=\frac{1}{2}[c_1x_1^2+c_2(x_2-x_1)^2]$$

Substituting these expressions into relation (6), Sec. 10, and performing the necessary operations indicated in formulas (5), Sec. 10, we obtain the differential equations of motion

$$\left.\begin{aligned} m_1\ddot{x}_1+(c_1+c_2)\,x_1-c_2x_2=0 \\ m_2\ddot{x}_2+c_2x_2-c_2x_1=0 \end{aligned}\right\} \tag{24}$$

where $a_1=m_1$; $a_2=m_2$; $a_{12}=0$; $k_1=c_1+c_2$; $k_2=c_2$; $k_{12}=-c_2$.

The system has only positional coupling. The coupling coefficient

$$\chi=\frac{-c_2}{\sqrt{(c_1+c_2)\,c_2}}=-\sqrt{\frac{c_2}{c_1+c_2}}$$

According to formula (18), the characteristic equation is

$$(c_1+c_2-m_1\Omega^2)(c_2-m_2\Omega^2)-c_2^2=0$$

or

$$m_1m_2\Omega^4-[(c_1+c_2)\,m_2+c_2m_1]\,\Omega^2+c_1c_2=0$$

Hence

$$\Omega_{1,\,2}^2=\frac{(c_1+c_2)\,m_2+c_2m_1\mp\sqrt{(c_1+c_2)^2\,m_2^2+c_2^2m_1^2-2c_2\,(c_1-c_2)\,m_1m_2}}{2m_1m_2} \tag{25}$$

The distribution coefficients are determined by formulas (20):

$$\alpha_1=\frac{c_1+c_2-m_1\Omega_1^2}{c_2}; \qquad \alpha_2=\frac{c_1+c_2-m_1\Omega_2^2}{c_2}$$

For the initial conditions $x_1=x_{10}$, $x_2=x_{20}$, $\dot{x}_1=\dot{x}_2=0$ at $t=0$ the solution of our simultaneous differential equations, in accordance with formulas (19), takes the form

$$x_1=A_1\cos\Omega_1t+A_2\cos\Omega_2t$$

$$x_2=A_1\alpha_1\cos\Omega_1t+A_2\alpha_2\cos\Omega_2t$$

where

$$A_1=-\frac{(c_1+c_2-m_1\Omega_2^2)\,x_{10}-c_2x_{20}}{m_1\,(\Omega_2^2-\Omega_1^2)}$$

$$A_2=\frac{(c_1+c_2-m_1\Omega_1^2)\,x_{10}-c_2x_{20}}{m_1\,(\Omega_2^2-\Omega_1^2)}$$

The normal coordinates x_1' and x_2' are determined, in accordance with (23), by the same relations as A_1 and A_2, but on their right-hand sides x_{10} and x_{20} must be replaced by the running values of the coordinates x_1 and x_2, respectively.

As a second example, consider a system (Fig. 24*b*) differing from the preceding one only in that the element *2* is connected to fixed stand *7* by spring *8* with a coefficient of stiffness c_3. We choose as general coordinates the same x_1 and x_2. The expression of the kinetic energy remains unchanged. The potential energy is now determined by the expression

$$\Pi = \frac{1}{2}[c_1 x_1^2 + c_2 (x_2 - x_1)^2 + c_3 x_2^2]$$

The differential equations of free vibrations become

$$m_1 \ddot{x}_1 + (c_1 + c_2) x_1 - c_2 x_2 = 0$$

$$m_2 \ddot{x}_2 + (c_3 + c_2) x_2 - c_2 x_1 = 0$$

To simplify matters, we assume $m_1 = m_2 = m$, $c_1 = c_3 = c$, $c_2 = c_0$ and rewrite the equations accordingly:

$$m\ddot{x}_1 + (c + c_0) x_1 - c_0 x_2 = 0$$

$$m\ddot{x}_2 + (c + c_0) x_2 - c_0 x_1 = 0$$

It can be seen that the coupling is purely positional. The coupling coefficient is

$$\chi_c = -\frac{c_0}{c + c_0}$$

The characteristic equation is

$$\begin{vmatrix} c + c_0 - m\Omega^2 & -c_0 \\ -c_0 & c + c_0 - m\Omega^2 \end{vmatrix} = 0$$

or

$$m^2\Omega^4 - 2(c + c_0) m\Omega^2 + c(c + 2c_0) = 0$$

Hence

$$\Omega_1 = \sqrt{\frac{c}{m}}; \quad \Omega_2 = \sqrt{\frac{c + 2c_0}{m}}$$

The distribution coefficients are $\alpha_1 = 1$ and $\alpha_2 = -1$. The general solution is represented by the relations

$$\left.\begin{aligned} x_1 &= A_1 \cos(\Omega_1 t - \varphi_1) + A_2 \cos(\Omega_2 t - \varphi_2) \\ x_2 &= A_1 \cos(\Omega_1 t - \varphi_1) - A_2 \cos(\Omega_2 t - \varphi_2) \end{aligned}\right\}$$

Thus, in the first normal mode of vibration the elements *1* and *2* vibrate in phase, in the second mode in opposite phase.

The normal coordinates

$$x_1' = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad x_2' = \frac{1}{2}(x_1 - x_2)$$

lend themselves readily to a physical interpretation: x_1' is the displacement of the centre of gravity of the system and x_2' is half the relative displacement of the elements *1* and *2* or the displacement of each relative to the centre of gravity of the system.

Assume now as initial conditions: $x_1 = 2a$; $x_2 = 0$; $\dot{x}_1 = \dot{x}_2 = 0$ at $t = 0$. The motion of our system will then be determined by the expressions

$$x_1 = a \cos \Omega_1 t + a \cos \Omega_2 t$$
$$x_2 = a \cos \Omega_1 t - a \cos \Omega_2 t$$

If $c_0 \ll c$, the natural frequencies Ω_1 and Ω_2 are close to each other and the free vibrations x_1 and x_2 will be of the nature of beats as illustrated in Fig. 24*c* (x_1 vibrations) and Fig. 24*d* (x_2 vibrations).

This example shows that with natural vibrations of multi-degree-of-freedom systems a periodic exchange of energy between the degrees of freedom is possible, the oscillations of one coordinate increasing at the expense of decreasing oscillations of another coordinate.

We shall now consider the forced vibrations of a two-degree-of-freedom conservative system in the case where both generalized forces Q_i in Eqs. (5), Sec. 10, are in-phase or antiphase sinusoidal functions $Q_1 \cos \omega t$ and $Q_2 \cos \omega t$ where $|Q_1|$ and $|Q_2|$ are the amplitudes of the forces. The equations of motion take the following form:

$$\left.\begin{aligned} a_1\ddot{q}_1 + a_{12}\ddot{q}_2 + k_1 q_1 + k_{12} q_2 &= Q_1 \cos \omega t \\ a_{12}\ddot{q}_1 + a_2\ddot{q}_2 + k_{12} q_1 + k_2 q_2 &= Q_2 \cos \omega t \end{aligned}\right\} \tag{26}$$

Let us assume the particular integrals corresponding to steady-state forced vibrations to have the form

$$q_1 = X_1 \cos \omega t, \quad q_2 = X_2 \cos \omega t \tag{27}$$

Inserting them into differential equations (26), we obtain two simultaneous equations linear in X_1 and X_2:

$$\left.\begin{aligned} X_1 (k_1 - a_1\omega^2) + X_2 (k_{12} - a_{12}\omega^2) &= Q_1 \\ X_1 (k_{12} - a_{12}\omega^2) + X_2 (k_2 - a_2\omega^2) &= Q_2 \end{aligned}\right\} \tag{28}$$

Solving (28), we obtain

$$X_1 = \frac{\begin{vmatrix} Q_1 & k_{12} - a_{12}\omega^2 \\ Q_2 & k_2 - a_2\omega^2 \end{vmatrix}}{\begin{vmatrix} k_1 - a_1\omega^2 & k_{12} - a_{12}\omega^2 \\ k_{12} - a_{12}\omega^2 & k_2 - a_2\omega^2 \end{vmatrix}}, \quad X_2 = \frac{\begin{vmatrix} k_1 - a_1\omega^2 & Q_1 \\ k_{12} - a_{12}\omega^2 & Q_2 \end{vmatrix}}{\begin{vmatrix} k_1 - a_1\omega^2 & k_{12} - a_{12}\omega^2 \\ k_{12} - a_{12}\omega^2 & k_2 - a_2\omega^2 \end{vmatrix}} \tag{29}$$

If the denominators of the right-hand sides of expressions (29) tend to zero, the absolute values of X_1 and X_2, generally speaking, increase infinitely, i.e., resonance sets in. By equating the denominators to zero, we obtain the characteristic equation (18) which is satisfied only when $\omega = \Omega_1$ and $\omega = \Omega_2$. Consequently, resonance can occur in a conservative system when the frequency of the exciting force is equal to one of the natural frequencies of the system.

As distinct from a system having one degree of freedom, special cases are possible in multi-degree-of-freedom systems when some of the resonances do not occur. This is the case where both the denominator and the numerator in the right-hand side of one of the two expressions (29) vanish simultaneously. The numerator of the other expression must then of necessity also be zero. Referring to expressions (20), we see that the numerators become zero when one of the following conditions is satisfied:

$$\frac{Q_1}{Q_2} = -\alpha_1 \text{ at } \omega=\Omega_1 \text{ and } \frac{Q_1}{Q_2} = -\alpha_2 \text{ at } \omega=\Omega_2$$

But, as has been pointed out earlier,

$$\alpha_1 = \frac{A_{21}}{A_{11}}, \quad \alpha_2 = \frac{A_{22}}{A_{12}}$$

where the first subscript stands for the order number of the generalized coordinate and the second for the order number of the natural frequency to which the given "amplitude"[1] of natural vibrations A pertains. Substituting the above values of α_1 and α_2 into condition (29), we obtain

$$\left.\begin{array}{l} Q_1A_{11} + Q_2A_{12} = 0 \text{ at } \omega=\Omega_1 \\ Q_1A_{21} + Q_2A_{22} = 0 \text{ at } \omega=\Omega_2 \end{array}\right\} \qquad (30)$$

If the "amplitudes" of free vibrations A_{11} and A_{12} are considered as components of one vector and the "amplitudes"[2] of the exciting forces Q_1 and Q_2 as components of another vector, then the first of the expressions (30) is the vector orthogonality condition of (Q_1, Q_2) and (A_{11}, A_{12}). The second equality expresses the vector orthogonality condition of (Q_1, Q_2) and (A_{21}, A_{22}). The physical interpretation is that the sum of the works performed by the exciting forces on the natural vibration displacements is zero (we are speaking here of cases where the frequency of the exciting forces are equal to one of the natural frequencies), i.e., in spite of the existence of exciting forces the system energy remains unchanged. Therefore the vibration amplitudes are limited and no resonance sets in.

[1] The word "amplitude" is placed in quotation marks because A_{ij} are equal to amplitudes accurate to the sign as A_{ij} can be either positive or negative.

[2] These can also be determined accurate to the sign.

Another important feature of multi-degree-of-freedom systems is that one of the numerators of the right-hand sides of expressions (29) can become zero at a frequency other than the natural frequencies of the system. At the same time the denominator is not zero. In this case the corresponding coordinate takes no part in the forced vibrations of the system and remains unchanged. A phenomenon occurs which is known as *antiresonance* or *dynamic absorbing* of vibrations[1]. A minimum on the amplitude response curve corresponds to the antiresonance and the amplitude value is zero in a conservative system. This phenomenon cannot be observed in single-degree-of-freedom systems.

We now turn to a brief consideration of dissipative systems. When dissipative forces appear in a system, the dissipative function (20), Sec. 10, for two degrees of freedom, takes the form

$$\Phi=\frac{1}{2}\left(b_1\dot{q}_1^2+2b_{12}\dot{q}_1\dot{q}_2+b_2\dot{q}_2^2\right) \tag{31}$$

The differential equations of free motion represented in the general case by expressions (24), Sec. 10, can be written as follows:

$$\left.\begin{aligned} a_1\ddot{q}_1+a_{12}\ddot{q}_2+b_1\dot{q}_1+b_{12}\dot{q}_2+k_1q_1+k_{12}q_2=0\\ a_{12}\ddot{q}_1+a_2\ddot{q}_2+b_{12}\dot{q}_1+b_2\dot{q}_2+k_{12}q_1+k_2q_2=0 \end{aligned}\right\} \tag{32}$$

The integral of linear equations with constant coefficients can always be taken to have the form

$$q_i=A_ie^{wt} \tag{33}$$

Substituting (33) into the left-hand sides of equations (32), we obtain two simultaneous homogeneous linear equations

$$\left.\begin{aligned} (a_1w^2+b_1w+k_1)A_1+(a_{12}w^2+b_{12}w+k_{12})A_2=0\\ (a_{12}w^2+b_{12}w+k_{12})A_1+(a_2w^2+b_2w+k_2)A_2=0 \end{aligned}\right\} \tag{34}$$

for which non-zero solutions are possible only if the condition

$$\begin{vmatrix} a_1w^2+b_1w+k_1 & a_{12}w^2+b_{12}w+k_{12}\\ a_{12}w^2+b_{12}w+k_{12} & a_2w^2+b_2w+k_2 \end{vmatrix}=0 \tag{35}$$

is satisfied.

Expression (35) represents a characteristic equation the solutions of which are four (in general, different) quantities w_1, w_2, w_3, w_4. Any of the simultaneous equations (34) yields the ratio $A_2 : A_1$, i.e., the distribution coefficients for the four roots w: α_1, α_2, α_3, α_4. The distribution coefficients corresponding to complex roots of the characteristic equation are also complex quantities. We can

[1] See Section 14.

now write the general solution of the simultaneous equations (32):

$$\left.\begin{aligned} q_1 &= A_1e^{w_1t} + A_2e^{w_2t} + A_3e^{w_3t} + A_4e^{w_4t} \\ q_2 &= A_1\alpha_1e^{w_1t} + A_2\alpha_2e^{w_2t} + A_3\alpha_3e^{w_3t} + A_4\alpha_4e^{w_4t} \end{aligned}\right\} \qquad (36)$$

The roots of the characteristic equation (35) can be real non-positive or complex in conjugated pairs with negative (or, in particular cases, zero) real parts. A particular solution of the type of one of the terms on the right-hand sides of (36) corresponds to each single real root. The sum of two particular solutions of type (51), Sec. 6, corresponds to two equal real roots. If the multiplicity of the root w_i exceeds two, for example, is s in a multi-degree-of-freedom system, the particular solution then takes the form

$$A_1e^{w_it} + A_2te^{w_it} + A_3t^2e^{w_it} + \ldots + A_st^{s-1}e^{w_it} \qquad (37)$$

The sum of two solutions of type (30), Sec. 6, corresponds to a pair of conjugate simple roots (not multiples). Let $w_1 = u + iv$ and $w_2 = u - iv$. According to expression (33) the sum of two particular solutions is

$$q_1 = A_1e^{w_1t} + A_2e^{w_2t}$$
$$q_2 = A_1\alpha_1e^{w_1t} + A_2\alpha_2e^{w_2t}$$

In this case the coefficients α_1 and α_2 will also be conjugate complex quantities: $\alpha_1 = \varkappa + i\lambda$, $\alpha_2 = \varkappa - i\lambda$. We take the arbitrary constants also in the form of conjugate complex quantities $A_1 = \frac{1}{2}(c_1 - ic_2)$, $A_2 = \frac{1}{2}(c_1 + ic_2)$. We may do so since only two constants, c_1 and c_2, are selected arbitrarily.

Substituting all the above values in our solution, we get:

$$q_1 = e^{ut}\left[\frac{1}{2}(c_1 - ic_2)e^{ivt} + \frac{1}{2}(c_1 + ic_2)e^{-ivt}\right] = e^{ut}(c_1\cos vt + c_2\sin vt)$$

The final transformation has been made using formula (8), Sec. 2.

$$q_2 = e^{ut}\left[\frac{1}{2}(c_1 - ic_2)(\varkappa + i\lambda)e^{ivt} + \frac{1}{2}(c_1 + ic_2)(\varkappa - i\lambda)e^{-ivt} = \right.$$
$$= e^{ut}[(c_1\varkappa + c_2\lambda)\cos vt + (c_2\varkappa - c_1\lambda)\sin vt]$$

Thus, the solution has been obtained in real form. In distinction to free vibrations of a conservative system, in the case considered the phase difference of oscillations of the coordinates is not equal to zero or π.

If the real part of a pair of conjugate roots of the characteristic equation is zero, i.e., they are imaginary, then the sum of two particular solutions takes the form (6), Sec. 6. If the conjugate roots are

of s-multiplicity (in a multi-degree-of-freedom system), the solution corresponding to $2s$ roots takes the form

$$q = e^{ut}(c_1 \cos vt + c_2 \sin vt + c_3 t \cos vt + c_4 t \sin vt + c_5 t^2 \cos vt + \\ + c_6 t^2 \sin vt + \ldots + c_{2s-1} t^{s-1} \cos vt + c_{2s} t^{s-1} \sin vt) \qquad (38)$$

With forced vibrations of a linear two-degree-of-freedom system in the case of a sinusoidal in-phase excitation the differential equations of motion can be written as follows:

$$\left.\begin{aligned} a_1\ddot{q}_1 + a_{12}\ddot{q}_2 + b_1\dot{q}_1 + b_{12}\dot{q}_2 + k_1 q_1 + k_{12} q_2 &= Q_1 \cos \omega t \\ a_{12}\ddot{q}_1 + a_2\ddot{q}_2 + b_{12}\dot{q}_1 + b_2\dot{q}_2 + k_{12} q_1 + k_2 q_2 &= Q_2 \cos \omega t \end{aligned}\right\} \qquad (39)$$

We assume the particular integral corresponding to steady-state forced vibrations to have a form analogous to that of expression (19), Sec. 7:

$$\left.\begin{aligned} q_1 &= A_1 \cos \omega t + B_1 \sin \omega t \\ q_2 &= A_2 \cos \omega t + B_2 \sin \omega t \end{aligned}\right\} \qquad (40)$$

In order to determine A_1, A_2, B_1, B_2, substitute expressions (40) in Eqs. (39) and equate the cosine and sine coefficients on the right- and left-hand sides of each of the identities obtained:

$$\begin{aligned} (k_1 - a_1\omega^2) A_1 + (k_{12} - a_{12}\omega^2) A_2 + b_1\omega B_1 + b_{12}\omega B_2 &= Q_1 \\ (k_{12} - a_{12}\omega^2) A_1 + (k_2 - a_2\omega^2) A_2 + b_{12}\omega B_1 + b_2\omega B_2 &= Q_2 \\ -b_1\omega A_1 - b_{12}\omega A_2 + (k_1 - a_1\omega^2) B_1 + (k_{12} - a_{12}\omega^2) B_2 &= 0 \\ -b_{12}\omega A_1 - b_2\omega A_2 + (k_{12} - a_{12}\omega^2) B_1 + (k_2 - a_2\omega^2) B_2 &= 0 \end{aligned}$$

The solution of these equations yields the required coefficients.

The forced oscillations of one coordinate are, in general, out of phase with respect to the oscillations of the other coordinate; this distinguishes dissipative systems from conservative ones. The second distinctive feature of dissipative systems is the dulling or complete absence of resonance phenomena, in a way similar to that pointed out in Sec. 7 for systems with one degree of freedom. The third distinctive feature is the dulling or complete absence of antiresonance even if there is any when dissipation is eliminated.

12. Continuous Systems (Systems with Distributed Parameters)

We have so far considered only discrete systems comprising separate sharply defined elements to each of which only one of the following properties is assigned: inertia (for instance, a point mass or undeformable solid body), elasticity (for instance, a spring), or

viscosity (for instance, a damper). We shall now briefly discuss the vibrations of some continuous systems whose inertia, elastic, and dissipative properties are distributed over the whole of the space occupied by the system. Every isolated element, no matter how small, possesses all the properties inherent in the system. Such systems are therefore said to be *systems with distributed parameters* as distinct from discrete systems whose parameters (mass, stiffness coefficient, resistance coefficient) are lumped in separate elements. Mechanical systems with distributed parameters are also called continuous media or continuums, thus laying stress on their characteristic property.

The knowledge of the state of a system with distributed parameters at a certain moment of time requires that the state at all its points at that moment be specified; the number of such points is infinite. A system with distributed parameters is sometimes said to have an infinite number of degrees of freedom. The mathematical means suitable for the description of such systems is a partial differential equation.

Systems with distributed parameters may be classified as one-dimensional, two-dimensional and three-dimensional. Real bodies have, of course, always three dimensions. However, in many cases when the extent of a body in one or two dimensions is predominant or when the state along one coordinate axis or two axes is sufficient to describe the behaviour of a system, this classification is justified.

A string and a rod can be cited as examples of one-dimensional systems, and a membrane and a thin plate as examples of two-dimensional ones.

It is characteristic of systems with distributed parameters that their vibrations involve wave propagation which in a mechanical system is the propagation of strains in the body (system). This can be illustrated by considering a string which is assumed to be an ideally flexible cord (in which no bending stresses originate) under tension. Let us impart to the string in section *123* a transverse deformation of the shape shown in Fig. 25*a*. This can be achieved by pulling the string laterally at point *2*, having preliminarily placed at points *1* and *3* two pegs which prevent deformation (lateral) of the rest of the string. We now let go the string at point *2* and pull out the pegs at points *1* and *3* simultaneously. Waves will be propagated along the string to the right and to the left.

The string sections at consecutive moments of time are represented in Fig. 25*b* by solid lines. Dotted lines show the auxiliary constructions.

In setting up the differential equation of the string transverse vibrations use will be made of the following assumptions: the string is ideally flexible and performs small transverse vibrations in one

plane; the mass of the string is uniformly distributed over its length; there is no energy dissipation.

In the equilibrium position the tensioned string coincided with the x-axis (Fig. 26). The string was then pulled laterally in the plane of the drawing in the y-direction and formed the curvilinear section shown in the figure. After being released at moment $t = 0$[1] the string started to vibrate freely.

The displacement of the string point with the abscissa x at time t is a function $y = y(x, t)$ whose form is as yet unknown. The force of string tension will be denoted by S. We now select the string section between points whose abscissas are x and $x + dx$ and set up the equation of motion using Newton's second law. If the linear density (mass per unit length) of the string is ρ_1, then the mass of the section selected is $\rho_1\, dx$. The acceleration of the section is $\partial^2 y/\partial t^2$.

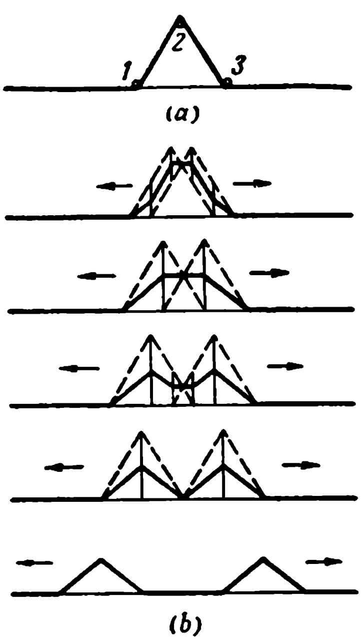


Figure 25

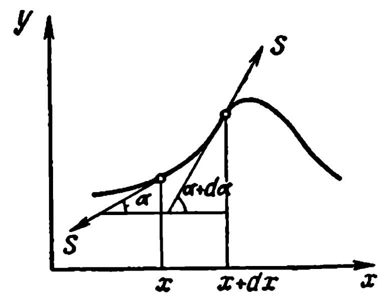


Figure 26

The sum of the projections of the tension forces onto the y-axis is equal to $-S \sin \alpha + S \sin(\alpha + d\alpha)$ where α is the angle of slope of the tangent line with respect to the abscissa. As the vibrations are small, we can assume $\sin \alpha = \alpha = \tan \alpha$. Using the first part of this equality, we obtain the projection of the tension forces in the form $S\, d\alpha$ and from the second part of the equality $Sd(\tan \alpha)$. Now $\tan \alpha = \partial y/\partial x$. Consequently we have

$$Sd\left(\frac{\partial y}{\partial x}\right) = S\,\frac{\partial^2 y}{\partial x^2}\,dx$$

[1] At this moment under the action of a momentary impulse distributed along the string its points can acquire initial velocities in the y-direction.

We can now write the differential equation of motion:

$$\rho_1 \frac{\partial^2 y}{\partial t^2} dx = S \frac{\partial^2 y}{\partial x^2} dx \tag{1}$$

Dividing both sides of expression (1) by $\rho_1 dx$, we obtain the wave equation in its final form[1]

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{2}$$

where

$$c = \sqrt{\frac{S}{\rho_1}} \tag{3}$$

Two classical techniques of solving the wave equation are known: *D'Alembert's method* (*travelling-wave solution*) and *Fourier's method* (*standing-wave solution*). The former is suitable in studying vibrations of an infinite as well as of a finite string; the latter can be used only to investigate the vibrations of a finite string. We shall first consider D'Alembert's method. We change the variables setting $y = y(u, v)$ where

$$u = x - ct, \quad v = x + ct \tag{4}$$

Applying the rule of composite function differentiation, we can write:

$$\left.\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= c^2 \left(\frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \, \partial v} + \frac{\partial^2 y}{\partial v^2} \right) \\ \frac{\partial^2 y}{\partial x^2} &= \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \, \partial v} + \frac{\partial^2 y}{\partial v^2} \end{aligned}\right\} \tag{5}$$

Substituting expressions (5) into the wave equation (2), we obtain

$$\frac{\partial^2 y}{\partial u \, \partial v} = 0 \quad \text{or} \quad \frac{\partial}{\partial u} \left(\frac{\partial y}{\partial v} \right) = 0 \tag{6}$$

As the derivative of the function $\partial y/\partial v$ with respect to u is zero, this function is independent of u and can only depend on v. So integrating Eq. (6) with respect to u, we obtain

$$\frac{\partial y}{\partial v} = f(v) \tag{7}$$

where $f(v)$ is an arbitrary function of v.

The integral of Eq. (7) with respect to v takes the form

$$y = \int f(v)\, dv + \varphi(u) \tag{8}$$

where $\varphi(u)$ is an arbitrary function of u.

---

[1] This form is taken also by the longitudinal vibration equation of a prismatic bar and of air or a liquid in a prismatic tube and by the torsional vibration equation of a cylindrical shaft.

Using the notation $\int f(v)\,dv = \psi(v)$, we now obtain

$$y = \varphi(u) + \psi(v) \tag{9}$$

Using expressions (4), we can now write the general solution as follows:

$$y = \varphi(x - ct) + \psi(x + ct) \tag{10}$$

This is known as *D'Alembert's integral.*

To elucidate the physical meaning of the solution obtained, consider first the particular case when $\psi = 0$ and assume that x is related to t by the expression

$$x - ct = k_1 = \text{const} \tag{11}$$

The derivative of x with respect to t obtained from (11)

$$\frac{dx}{dt} = c \tag{12}$$

It can be asserted now that the relation (11) determines the abscissa x of a point that moves at the velocity c in the positive direction of the abscissa axis. With this relation $y = \varphi(k_1) = \text{const}$. Hence the point moving at the velocity c in the positive direction of the abscissa axis is the one whose ordinate y remains constant and is equal to $\varphi(k_1)$ at all times. Thus c is the velocity of propagation of the wave along the string in the positive direction (direct wave), and the particular solution $y = \varphi(x - ct)$ corresponds to the travelling direct wave.

With $\varphi = 0$ and $x + ct = k_2 = \text{const}$, it can be seen that the particular solution $\psi(x + ct)$ corresponds to the inverse wave travelling in the negative direction of the abscissa axis at the velocity $-c$.

The arbitrary functions φ and ψ can be determined from the initial conditions at $t = 0$:

$$y = f(x), \quad \frac{\partial y}{\partial t} = \Phi(x) \tag{13}$$

Inserting the first of the conditions into expression (10), we obtain

$$f(x) = \varphi(x) + \psi(x) \tag{14}$$

Differentiating expression (10) with respect to time and substituting into it the second condition, we obtain

$$\Phi(x) = -c\varphi'(x) + c\psi'(x)$$

and upon integrating with respect to x we have

$$\frac{1}{c}\int_0^x \Phi(\xi)\,d\xi = -\varphi(x) + \psi(x) \tag{15}$$

From the simultaneous equations (14) and (15) we determine

$$\varphi(x)=\frac{1}{2}f(x)-\frac{1}{2c}\int_0^x \Phi(\xi)\,d\xi$$

$$\psi(x)=\frac{1}{2}f(x)+\frac{1}{2c}\int_0^x \Phi(\xi)\,d\xi$$

Passing from the initial time value $t=0$ to the running time variable, i.e. using the argument $x-ct$ instead of x in the first equality and the argument $x+ct$ in the second one and substituting the values of φ and ψ into the general integral (10), we can write the following particular solution satisfying the initial conditions (13):

$$y=\frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\Phi(\xi)\,d\xi \qquad (16)$$

Two waves are propagated along an infinite string—to the right and to the left (see Fig. 25*b*). If the string is of a finite length, the conditions at the ends must be specified. With fixed string ends the boundary conditions take the form

$$y=0 \quad \text{at} \quad x=0,\ y=0 \quad \text{at} \quad x=l \qquad (17)$$

where l is the string length (the right-end abscissa).

Having reached the string end (the point of rigid fixation) the wave is reflected and begins moving in the opposite direction; the sign of the deformation is reversed. The wave reflection is shown in Fig. 27*a*.

Consider the particular case of the vibrations of a fixed string with the end conditions (17) and the initial conditions at $t=0$:

$$y=y_{max}\sin\frac{\pi}{l}x,\quad \frac{\partial y}{\partial t}=0$$

This means that at the initial moment the shape of the string is the half-wave of a sine curve. Substituting the y-values into the solution (16), we obtain

$$y=\frac{1}{2}y_{max}\left[\sin\frac{\pi}{l}(x-ct)+\sin\frac{\pi}{l}(x+ct)\right]$$

Hence

$$y=y_{max}\sin\frac{\pi}{l}x\cos\frac{\pi c}{l}t \qquad (18)$$

We have obtained the equation of the standing wave whose period of vibration

$$\tau=\frac{2l}{c} \qquad (19)$$

The character of the string vibrations is presented in Fig. 27*b*. We now turn to Fourier's method. The solution of the wave equation is sought in the form of the product of two functions of which

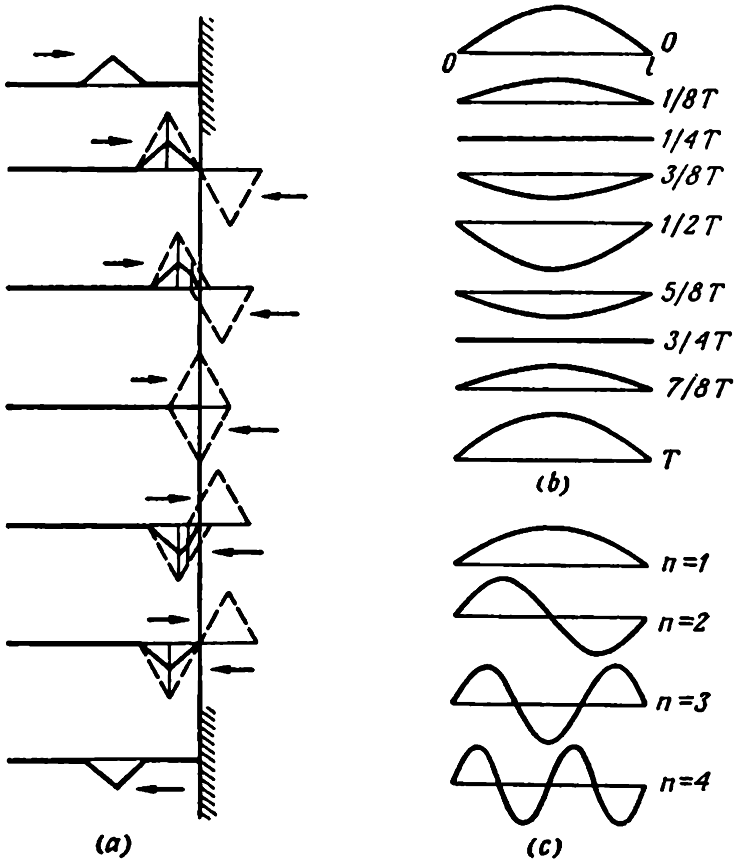


Figure 27

one depends only on t and the other only on x:

$$y = T(t)\,X(x) \tag{20}$$

Substituting this solution into Eq. (2), we obtain

$$\frac{d^2T}{dt^2}X = c^2\frac{d^2X}{dx^2}T$$

Separation of the variables leads to

$$\frac{1}{c^2T}\cdot\frac{d^2T}{dt^2} = \frac{1}{X}\cdot\frac{d^2X}{dx^2}$$

The left-hand side of the equation depends only on t, and the right-hand side only on x. The two sides can be equal if they

are constants; let us denote this constant by $-\lambda^2$. Then.

$$\frac{1}{c^2T}\cdot\frac{d^2T}{dt^2}=\frac{1}{X}\cdot\frac{d^2X}{dx^2}=-\lambda^2$$

and we obtain from this expression two ordinary linear differential equations:

$$\frac{d^2T}{dt^2}+c^2\lambda^2T=0$$

$$\frac{d^2X}{dx^2}+\lambda^2X=0$$

Their general solutions take the form (6), Sec. 6:

$$T=A\cos c\lambda t+B\sin c\lambda t$$

$$X=C\cos\lambda x+D\sin\lambda x$$

Inserting them into the solution (20), we can write the general integral of the wave equation (2):

$$y=(A\cos c\lambda t+B\sin c\lambda t)(C\cos\lambda x+D\sin\lambda x) \tag{21}$$

To determine the constants C, D and λ we now make use of the boundary conditions (17) which yield

$$(A\cos c\lambda t+B\sin c\lambda t)\,C=0 \tag{22}$$

$$(A\cos c\lambda t+B\sin c\lambda t)(C\cos\lambda l+D\sin\lambda l)=0 \tag{23}$$

The first factor in equality (22) may be zero only when $A=B=0$ but the corresponding solution is $y\equiv 0$ which corresponds to a state of equilibrium and does not imply vibrations. Therefore we can only make the assumption $C=0$. Then from (23) we have

$$\sin\lambda l=0 \tag{24}$$

since with $D=0$ we obtain again $y\equiv 0$.

Equation (24) is satisfied by an infinite number of λ values:

$$\lambda_n=\frac{n\pi}{l},\quad (n=1,\ 2,\ \ldots)^1 \tag{25}$$

We now write the particular integral corresponding to the nth value of λ by substituting this value from expression (25) into the general solution (21):

$$y_n=\left(a_n\cos\frac{n\pi c}{l}t+b_n\sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi}{l}x$$

[1] When $n=0$, we obtain again the trivial solution $y\equiv 0$.

The sum of all the particular integrals is the general solution corresponding to any initial conditions:

$$y=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi c}{l}t+b_n\sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi}{l}x \tag{26}$$

To make the solution (26) satisfy the initial conditions (13) it remains only to determine the arbitrary constants a_n and b_n. From the first initial condition

$$f(x)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi}{l}x$$

Taking the derivative of the integral (26) and using the second of the initial conditions, we obtain

$$\Phi(x)=\sum_{n=1}^{\infty}\frac{n\pi c}{l}b_n\sin\frac{n\pi}{l}x$$

The last two expressions are expansions of the initial functions $f(x)$ and $\Phi(x)$ in a Fourier series in terms of sines[1]. Therefore from formula (2), Sec. 4, we have[2]

$$a_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$$

$$b_n=\frac{2}{n\pi c}\int_0^l \Phi(x)\sin\frac{n\pi}{l}x\,dx \tag{27}$$

It is readily seen that the solution for the string whose shape is the half-wave of a sine curve and whose velocities are zero at the initial moment is expression (18) which has been obtained above by using D'Alembert's method.

The expressions $\sin\pi x/l$, $\sin 2\pi x/l$ and $\sin 3\pi x/l$ represent the normal elastic curves. Several normal elastic curves are shown in Fig. 27*c*.

The natural frequency

$$\Omega_n=\frac{n\pi c}{l},\quad (n=1,2,3,\ldots) \tag{28}$$

and the period of vibration

$$\tau_n=\frac{2l}{nc} \tag{29}$$

correspond to each normal elastic curve.

---

[1] The expansion is valid only in the domain where the function is defined, i.e., within the interval $0\leqslant x\leqslant l$.

[2] We arbitrarily define $f(x)$ and $\Phi(x)$ on the additional segment $-l\leqslant x\leqslant 0$ as odd functions.

Consequently, a system with distributed parameters has an infinite number of natural frequencies. Various particular cases of string vibrations are superpositions of n normal elastic curves of natural vibrations. That is why a vibrating string generally emits, apart from the fundamental (first) tone, a number of overtones (higher tones).

If there is a uniformly distributed linear damping, the wave equation takes the form

$$\frac{\partial^2 y}{\partial t^2}+2h\frac{\partial y}{\partial t}=c^2\frac{\partial^2 y}{\partial x^2} \tag{30}$$

With a continuously distributed exciting force the equation of motion assumes the following form:

$$\frac{\partial^2 y}{\partial t^2}+2h\frac{\partial y}{\partial t}=c^2\frac{\partial^2 y}{\partial x^2}+\frac{q(x,t)}{\rho_1} \tag{31}$$

where $q(x, t)$ = external force per unit length
h = damping coefficient.

Figure 28

Kinematic excitation can be represented by suitable end conditions.

Consider the free longitudinal vibrations of a thin prismatic bar, an element of which is pictured in Fig. 28. We shall neglect the dissipation of energy accompanying the vibrations and the cross-sectional Poisson contraction. Let x denote the coordinate measured along the bar, y the elastic deformation in the same direction; y is a function of the coordinate x and time t, i.e., $y = y(x, t)$. The form of this function is to be determined.

Consider a bar element of length dx as shown in Fig. 28. An elastic force S is applied to the left cross-section, and the force $S + dS$ to the right cross-section of the bar. The mass of the element

$$dm=\rho F\,dx$$

where ρ = density per unit volume
F = cross-sectional area of the bar.

The acceleration of the bar element is $\partial^2 y/\partial t^2$. The elastic force

$$S=EF\frac{\partial y}{\partial x}$$

where E = modulus of elasticity (Young's modulus)

$\frac{\partial y}{\partial x}$ = relative strain.

Hence the resultant of the forces applied to the element of the bar

$$dS = EF \frac{\partial^2 y}{\partial x^2} dx$$

The differential equation of motion of the element in question can be represented in the form

$$\rho F \frac{\partial^2 y}{\partial t^2} dx = EF \frac{\partial^2 y}{\partial x^2} dx \qquad (32)$$

Hence, using the notation

$$c = \sqrt{\frac{E}{\rho}}$$

we obtain the wave equation (2):

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (33)$$

The general solution of this equation for a bar of length l takes the form of Eq. (21):

$$y = (A \cos c\lambda t + B \sin c\lambda t)(C \cos \lambda x + D \sin \lambda x)$$

The conditions at the fixed ends of the bar have the form $y = 0$ given by expression (17).

The end conditions for a bar with free ends are determined by the expression

$$\frac{\partial y}{\partial x} = 0$$

which follows from the condition that the elastic force at the free end is zero.

If both ends of the bar are fixed, the end conditions are determined by (17), whence $C = 0$ and λ is determined by expression (25):

$$\lambda_n = \frac{n\pi}{l}, \quad (n = 1, 2, \ldots)$$

To each value of λ_n there correspond the normal elastic curve

$$X_n = \sin \frac{n\pi}{l} x \qquad (34)$$

and the natural frequency

$$\Omega_n = \frac{n\pi c}{l} \qquad (35)$$

For a bar with free ends the end conditions can be written as follows:

$$\frac{\partial y}{\partial x} = 0 \text{ at } x = 0, \quad \frac{\partial y}{\partial x} = 0 \text{ at } x = l \qquad (36)$$

Differentiating expression (21) with respect to x and using the end conditions (36), we obtain

$$(A \cos c\lambda t + B \sin c\lambda t)\,\lambda D = 0$$

$$(A \cos c\lambda t + B \sin c\lambda t)\,\lambda\,(-C \sin \lambda l + D \cos \lambda l) = 0$$

from which $D = 0$ and $\sin \lambda l = 0$, and consequently

$$\lambda_n = \frac{n\pi}{l}, \quad (n = 0, 1, 2, \ldots)$$

which is similar to the preceding case and to expression (25), differing in that there is a root $\lambda_0 = 0$ at $n = 0$, corresponding to the uniform motion of the bar as a rigid body.

The normal mode shapes are described by the expressions

$$X_n = \cos \frac{n\pi}{l} x \tag{37}$$

and the natural frequencies

$$\Omega_n = \frac{n\pi c}{l} \tag{38}$$

This is analogous to expression (35) in the preceding case; when $n = 0$ we have $\Omega_0 = 0$ which corresponds to the above-mentioned motion of the bar as a rigid body.

If the left end of the bar is fixed and the right one free, the end conditions are determined by the expressions

$$y = 0 \text{ at } x = 0, \quad \frac{\partial y}{\partial x} = 0 \text{ at } x = l \tag{39}$$

We obtain accordingly

$$(A \cos c\lambda t + B \sin c\lambda t)\,C = 0$$

$$(A \cos c\lambda t + B \sin c\lambda t)\,\lambda\,(-C \sin \lambda l + D \cos \lambda l) = 0$$

whence $C = 0$ and $\cos \lambda l = 0$; consequently

$$\lambda_n = \left(n + \frac{1}{2}\right)\frac{\pi}{l}, \quad (n = 0, 1, 2, \ldots) \tag{40}$$

The normal mode shapes are given by the expressions

$$X_n = \sin \left(n + \frac{1}{2}\right)\frac{\pi}{l} x \tag{41}$$

and the natural frequencies by

$$\Omega_n = \left(n + \frac{1}{2}\right)\frac{\pi c}{l} \tag{42}$$

Figure 29 illustrates the first few normal mode shapes of longitudinally vibrating bars with both ends fixed (*a*), both ends free (*b*), left end fixed and right end free (*c*). The curves are a conventional

representation of the deformations which in fact are directed along the bar and not laterally.

In all the cases considered the equation of vibrations is

$$y = \sum_{n=0 \text{ or } 1}^{\infty} (a_n \cos \Omega_n t + b_n \sin \Omega_n t) X_n$$

where X_n are called *eigenfunctions* or *fundamental functions.*

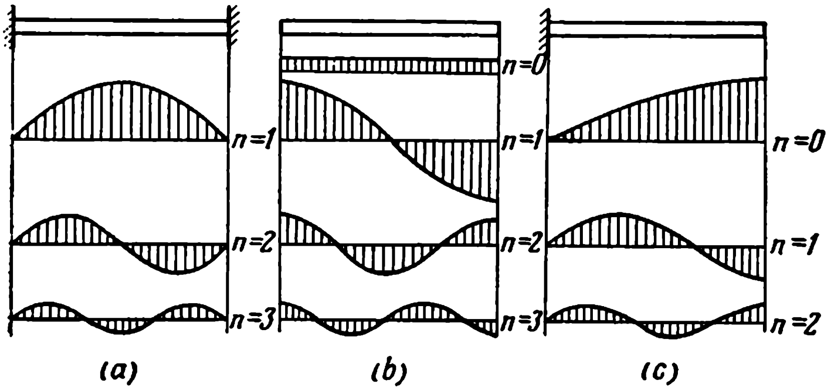


Figure 29

The coefficients a_n and b_n are determined from the initial conditions (13).

We now turn to the free plane lateral (bending) vibrations of thin prismatic bars (beams) in the direction of one of the principal axes of stiffness. We shall neglect the dissipation of energy in vibrations, and the part played by longitudinal and shearing forces and by the inertia of the turning cross-section. Denote the coordinate measured along the beam by x, and the transverse deformation by y. The equation of the axis of the beam bent by a uniformly distributed load q can be written in the form

$$EI \frac{\partial^4 y}{\partial x^4} = q \tag{43}$$

where E = Young's modulus

I = (equatorial) moment of inertia of the cross-section of the beam.

For free vibrations the load q is due only to inertia, i.e.,

$$q = -\rho_1 \frac{\partial^2 y}{\partial t^2} \tag{44}$$

where ρ_1 is the linear density of the beam (mass per unit length).

It follows that the differential equation of the motion of the beam will be

$$\frac{\partial^4 y}{\partial x^4} = -\frac{\rho_1}{EI} \cdot \frac{\partial^2 y}{\partial t^2} \tag{45}$$

For a beam of length l the solution is sought by using Fourier's method in the form of a standing wave:

$$y = X(x)\,T(t) \tag{46}$$

Substituting this solution into the differential equation (45), we obtain

$$-\frac{\rho_1}{EIT}\cdot\frac{d^2T}{dt^2} = \frac{1}{X}\cdot\frac{d^4X}{dx^4}$$

Both sides of this equation must be equal to a constant which we denote by λ^4. From this it follows that

$$\frac{d^2T}{dt^2} + \Omega^2 T = 0 \tag{47}$$

$$\frac{d^4X}{dx^4} - \lambda^4 X = 0 \tag{48}$$

where

$$\Omega = \lambda^2 \sqrt{\frac{EI}{\rho_1}} \tag{49}$$

We can now write the general solutions of Eqs. (47) and (48):

$$T = A\cos\Omega t + B\sin\Omega t \tag{50}$$

$$X = C_1\cos\lambda x + C_2\sin\lambda x + C_3\cosh\lambda x + C_4\sinh\lambda x \tag{51}$$

Before specifying the end conditions for determining the integration constants of the solution (51) note that the slope angle of the axis of the bent beam (for sufficiently small deflections)

$$\theta = \frac{\partial y}{\partial x}$$

the bending moment in the cross-section of the beam

$$M = EI\frac{\partial^2 y}{\partial x^2}$$

and the shearing force in the cross-section

$$Q = EI\frac{\partial^3 y}{\partial x^3}$$

If one end of the beam is free (Fig. 30*a*), the end conditions are determined by equating the bending moment and the shearing force to zero; hence

$$\frac{d^2X}{dx^2} = 0, \quad \frac{d^3X}{dx^3} = 0$$

If the end of the beam is connected to the support by a hinge (Fig. 30*b*), the end conditions are determined by equating the

deflection and the bending moment to zero; hence

$$X=0, \quad \frac{d^2X}{dx^2}=0$$

With a fixed beam end (Fig. 30c), the end conditions are determined by equating the deflection and the slope of the axis of the bent beam to zero; hence

$$X=0, \quad \frac{dX}{dx}=0$$

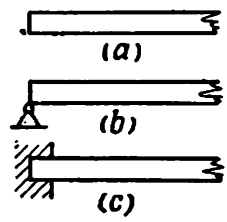


Figure 30

Using the conditions at both ends, we obtain simultaneous homogeneous linear equations in the integration constants. The condition for obtaining a solution other than the trivial one, viz. $C_1 = C_2 = C_3 = C_4 = 0$, is that the determinant of the equations be equal to zero, which leads to a transcendental (or trigonometric if both ends are supported on hinges) equation in λ. This equation has an infinite number of non-negative roots λ_n which determine the fundamental functions X_n and the natural frequencies Ω_n.

In the simplest case when both ends of the beam are supported on hinges, the conditions at the left end yield the equations

$$C_3+C_1=0, \quad C_3-C_1=0$$

Hence $C_1=0$, $C_3=0$. Using this result, the conditions at the right end lead to the equations

$$C_4 \sinh \lambda l + C_2 \sin \lambda l = 0$$
$$C_4 \sinh \lambda l - C_2 \sin \lambda l = 0$$

Since the solution $C_2=C_4=0$ satisfies the conditions of equilibrium rather than those of motion, the determinant of the last two equations must be equal to zero:

$$2 \sinh \lambda l \sin \lambda l = 0$$

As a result, we have $\sin \lambda l = 0$ and

$$\lambda_n = \frac{n\pi}{l}, \quad (n=1, 2, \ldots) \tag{52}$$

and, consequently, $C_4=0$.

We can now write the expressions for the fundamental functions:

$$X_n = \sin \frac{n\pi}{l} x \tag{53}$$

and for the natural frequencies:

$$\Omega_n = \frac{n^2\pi^2}{l^2} \sqrt{\frac{EI}{\rho_1}} \tag{54}$$

The equation of vibrations takes the form

$$y=\sum_{n=1}^{\infty}(a_n\cos\Omega_n t+b_n\sin\Omega_n t)\,X_n \tag{55}$$

where a_n and b_n are determined by the initial conditions.

The normal mode shapes of vibrating beams correspond to those shown in Fig. 29a with the difference that in this case the graphs represent actual deformations. Note the difference between the two sets of frequencies in these two cases.

For longitudinal vibrations of a bar fixed at both ends we have from formula (35) $\Omega_1:\Omega_2:\Omega_3=1:2:3$, and for transverse vibrations of a beam supported on hinges at both ends, from formula (54), $\Omega_1:\Omega_2:\Omega_3=1^2:2^2:3^2=1:4:9$.

We have discussed only cases of free vibrations of one-dimensional systems with distributed parameters. The vibrations of two-dimensional systems (membranes, plates) are described by differential equations containing three arguments (time and two coordinates). The vibrations of three-dimensional systems depend on four arguments—time and three coordinates.

The most important feature of systems with distributed parameters is that they have an infinite number of natural frequencies of vibrations and of corresponding shapes of the normal modes of vibration.

The free vibrations of such systems are a superposition of normal mode shapes, which is determined by the initial conditions. If the initial conditions correspond to one of the normal mode shapes, the system vibrates only at one frequency corresponding to this mode shape.

Systems with distributed parameters performing forced vibrations may have an infinite number of resonances since they have an infinite set of natural frequencies.

13. Amplitude and Phase Response Curves

To obtain generalized results and minimize the number of parameters that determine the behaviour of a system it is reasonable to introduce dimensionless quantities; the particular set of dimensionless quantities selected depends on the problem to be solved. Thus, for the system described by differential equation (17), Sec. 7, we can select as dimensionless variables the argument τ and the function ξ defined by the relations

$$\tau=\omega_0 t,\quad \xi=\frac{m\omega_0^2}{F_n}x \tag{1}$$

Not infrequently the quantities τ and ξ are called, respectively, the dimensionless time and the dimensionless displacement. These

quantities are proportional to the corresponding dimensional variables. However, in using these and similar terms one must have in mind their conventional nature.

Introducing the new variables into the differential equation (17), Sec. 7, it takes the form

$$\ddot{\xi} + 2\beta\dot{\xi} + \xi = \cos\gamma\tau \tag{2}$$

where the dimensionless variables β and γ are defined by the expressions[1]

$$\beta = \frac{h}{\omega_0}, \quad \gamma = \frac{\omega}{\omega_0} \tag{3}$$

Here and further in similar cases the dots above the function notation are used to denote differentiation with respect to the corresponding dimensionless argument.

Other dimensionless variables can be introduced, viz.,

$$\tau_* = \omega t, \quad \xi_* = \frac{m\omega^2}{F_a}x \tag{4}$$

whose substitution into Eq. (17), Sec. 7, transforms the latter into the following form:

$$\ddot{\xi}_* + 2\beta_*\dot{\xi}_* + \gamma_*^2\xi_* = \cos\tau_* \tag{5}$$

where

$$\beta_* = \frac{h}{\omega}, \quad \gamma_* = \frac{\omega_0}{\omega} \tag{6}$$

are dimensionless variables that determine the behaviour of the system.

The change from the first set of dimensionless quantities to the second one is represented by the relations

$$\tau_* = \tau\gamma, \quad \xi_* = \xi\gamma^2, \quad \beta_* = \frac{\beta}{\gamma}, \quad \gamma_* = \frac{1}{\gamma} \tag{7}$$

These relations are invariant in respect of the direction of transformation of the dimensionless quantities: the change from the second to the first set is determined by analogous relations:

$$\tau = \tau_*\gamma_*, \quad \xi = \xi_*\gamma_*^2, \quad \beta = \frac{\beta_*}{\gamma_*}, \quad \gamma = \frac{1}{\gamma_*} \tag{8}$$

The particular integral of the differential equation (2), corresponding to steady forced vibrations, takes the following form for the first set of parameters

$$\xi = \xi_a \cos(\gamma\tau - \varphi) \tag{9}$$

---

[1] The parameter β (damping ratio) was discussed in Sec. 6 [see, for instance, formula (43), Sec. 6].

where

$$\xi_a = \frac{1}{\sqrt{(1-\gamma^2)^2 + 4\beta^2\gamma^2}} \tag{10}$$

$$\varphi = \tan^{-1}\frac{2\beta\gamma}{1-\gamma^2} \tag{11}$$

The corresponding particular integral of Eq. (5) can be written as follows:

$$\xi_* = \xi_{*a}\cos(\tau_* - \varphi) \tag{12}$$

where

$$\xi_{*a} = \frac{1}{\sqrt{(\gamma_*^2 - 1)^2 + 4\beta_*^2}} \tag{13}$$

$$\varphi = \tan^{-1}\frac{2\beta_*}{\gamma_*^2 - 1} \tag{14}$$

Note that formulas (21), Sec. 7, (11) and (14) determine the same phase shift φ.

Thus the behaviour of a single-degree-of-freedom system excited by a sinusoidally varying force is determined by two dimensionless parameters. The character of the natural motion of a single-degree-of-freedom system is determined by one dimensionless parameter.

The dimensionless response curves can be plotted using γ or γ_* values on the abscissa. In the latter case formula (10) can be written, for instance, in the form

$$\xi_a = \frac{\gamma_*^2}{\sqrt{(\gamma_*^2 - 1)^2 + 4\beta_*^2}}$$

When γ or γ_* are laid off along the abscissa, the graphical representation of the response curves is not convenient in some cases as both arguments change within the limits from zero to infinity. As the zero value of one of these parameters corresponds to the infinitely large value of the other, we shall plot, for the purpose of representing the whole of the response curve on a finite segment of the abscissa, the first part of the curve by laying off γ values in the segment $0 \leqslant \gamma \leqslant 1$ along the abscissa from left to right (as is usually done); this segment corresponds to the interval $\infty > \gamma_* \geqslant 1$. The second part of the curve will be plotted on another graph by laying off γ_* values in the segment $0 \leqslant \gamma_* \leqslant 1$ along the abscissa from right to left (i.e., in the opposite direction); this segment corresponds to the interval $\infty > \gamma \geqslant 1$. The ordinates of both graphs are plotted to the same scale. By sewing the ends of the graphs at the argument values of $\gamma = \gamma_* = 1$ we obtain a full response curve.

To avoid plotting the curves in opposite directions for two different arguments we shall introduce a new argument:

$$\sigma=\begin{cases} \gamma & \text{at} \quad 0\leqslant\gamma\leqslant 1 \\ 2-\dfrac{1}{\gamma} & \text{at} \quad 1\leqslant\gamma<\infty \end{cases} \tag{15}$$

laying it off along the abscissa, as usual, from left to right. The result is the same graph as that obtained by sewing the ends of the two half-curves. The new argument varies in the finite segment $0 \leqslant \sigma \leqslant 2$.

An analogous graph can be plotted by laying off along the abscissa from right to left the argument

$$\sigma_*=\begin{cases} \gamma_* & \text{at} \quad 0\leqslant\gamma_*\leqslant 1 \\ 2-\dfrac{1}{\gamma_*} & \text{at} \quad 1\leqslant\gamma_*<\infty \end{cases} \tag{16}$$

Consider the physical meaning of the dimensionless quantities ξ and ξ_*. The second of the formulas (1) shows that ξ is the ratio of the running value of displacement x to x_0[1]:

$$x_0=\frac{F_a}{m\omega_0^2}=\frac{F_a}{c}=x_{st} \tag{17}$$

This is the displacement amplitude x_0 at $\omega\to 0$ which is equal to the static deformation x_{st} of the spring under the action of the constant force F_a. The dimensionless amplitude

$$\xi_a=\frac{x_a}{x_{st}} \tag{18}$$

is often called the *dynamic response factor*.

The second of formulas (4) shows that ξ_* is the ratio of the running value of the displacement x to

$$x_\infty=\frac{F_a}{m\omega^2} \tag{19}$$

which is the displacement amplitude at $h=0$ and $\gamma=\infty$ ($\omega_0=0$), i.e., the displacement amplitude of a free body of mass m (not connected with spring and damper) under the action of the force $F_a \cos \omega t$. Consequently, the dimensionless displacement amplitude

$$\xi_{*a}=\frac{x_a}{x_\infty} \tag{20}$$

We now introduce one more dimensionless quantity — the *magnification factor*

$$\xi_m=\frac{x_a}{x_t}=\sqrt{\frac{1+4\beta_*^2}{(\gamma_*^2-1)^2+4\beta_*^2}}=\frac{\gamma\sqrt{\gamma^2+4\beta^2}}{\sqrt{(1-\gamma^2)^2+4\beta^2\gamma^2}} \tag{21}$$

[1] The second of equalities (17) follows from expression (5), Sec. 6. The x_{st} in (17) should not be confused with x_{st} in expression (15), Sec. 6.

which is the ratio of the amplitude of forced vibrations x_a of our system described by Eq. (16), Sec. 7, to the amplitude of forced vibrations x_t of a truncated system having no spring and described by the differential equation $m\ddot{x} + b\dot{x} = F_a \cos \omega t$. The magnification factor is useful in characterizing systems operating at resonance or near it.

With comparatively small values of β use may be made of the following approximate formulas for the resonance magnification factor $\xi_{m\ res}$ at $\gamma = 1$:

$$\xi_{m\,res} \approx \frac{1}{2\beta}, \quad \xi_{m\,res} \approx \frac{1}{2\beta} + \beta \tag{22}$$

and the error in using the first at $\beta < 0.22$ and the second at $\beta < 0.6$ will not exceed 10%.

Figure $31a$[1] illustrates the dimensionless displacement amplitude response curves $\xi_a(\sigma)$ for several values of the damping ratio β plotted according to formula (10). At $\gamma_m = \sigma_m = \sqrt{1 - 2\beta^2}$ the curves show a maximum

$$\xi_{a\,max} = \frac{1}{2\beta\sqrt{1-\beta^2}} \tag{23}$$

Obviously the maxima exist only with damping ratios β less than the limiting value $\beta_l = 1/\sqrt{2}$.

Figure $31b$ shows amplitude response curves of the dimensionless vibration velocity $\dot{\xi}_a(\sigma)$ plotted according to the expression $\dot{\xi}_a = \xi_a \gamma$ which follows from the solution (9). When $\gamma_m = \sigma_m = 1$, the curves have maxima

$$\dot{\xi}_{a\,max} = \frac{1}{2\beta} \tag{24}$$

for any value of β.

The amplitude response curves of the dimensionless acceleration $\ddot{\xi}_a(\sigma)$ (Fig. $31c$) are plotted on the basis of the equality $\ddot{\xi}_a = \xi_a \gamma^2$. With $\gamma_m = \frac{1}{\sqrt{1-2\beta^2}}$, i.e., $\sigma_m = 2 - \sqrt{1 - 2\beta^2}$, the curves have maxima

$$\ddot{\xi}_{a\,max} = \xi_{a\,max} \tag{25}$$

which exist only with β values less than the above limiting value.

We have seen that with $0 < \beta < 1/\sqrt{2}$ the maxima, i.e., the resonant values of ξ_a, $\dot{\xi}_a$, $\ddot{\xi}_a$ correspond to different γ values.

---

[1] In Figs. 31, $32b$, $33a$, 34, 36-44 the curves for $\beta = 0$ are marked by *1*, or $\beta = 0.1$ by *2*, for $\beta = 0.25$ by *3*, for $\beta = \sqrt{2}/2$ by *4*, for $\beta = 1$ by *5*, and or $\beta = 2$ by *6*.

With the argument selected, σ, the curves representing $\dot{\xi}_a(\sigma)$ are symmetrical with respect to the straight line $\sigma = 1$. The curves $\xi_a(\sigma)$ and $\ddot{\xi}_a(\sigma)$ are symmetrical to one another with respect to the same line.

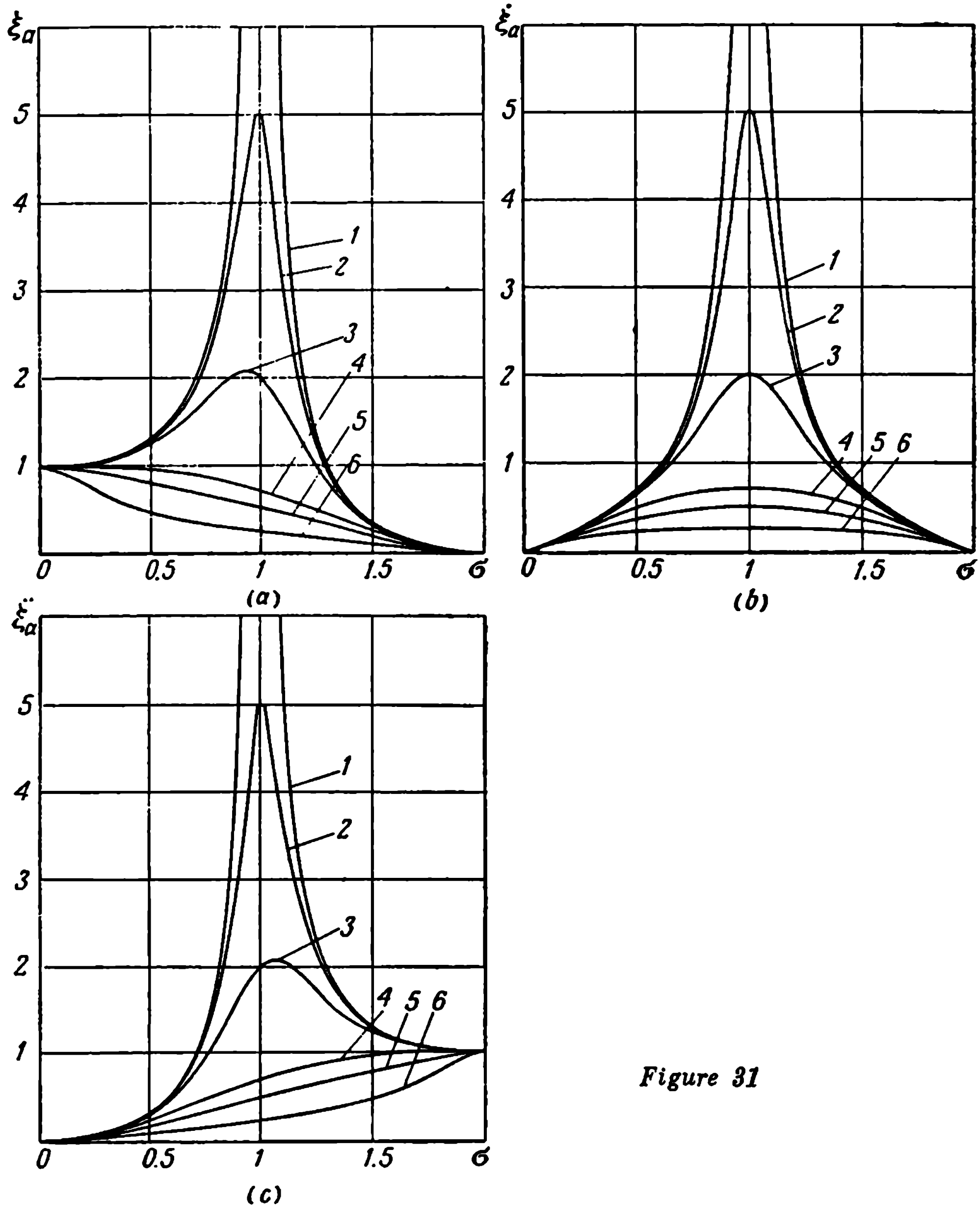


Figure 31

The response curves $\xi_{*a}(\sigma) = \dot{\xi}_{*a}(\sigma) = \ddot{\xi}_{*a}(\sigma)$ plotted according to relation (13) for several β_* values are shown in Fig. 32a[1].

[1] In Figs. 32a and 33b the curves for $\beta_* = 0$ are marked by *1*, for $\beta_* = 0.25$ by *2*, for $\beta_* = 1$ by *3*, and for $\beta_* = 2$ by *4*.

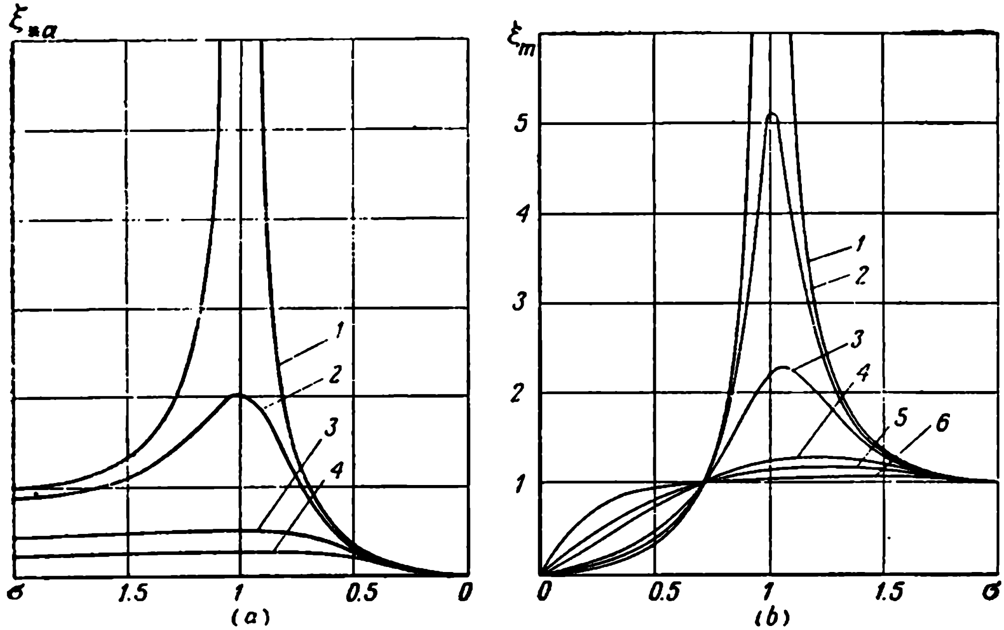


Figure 32

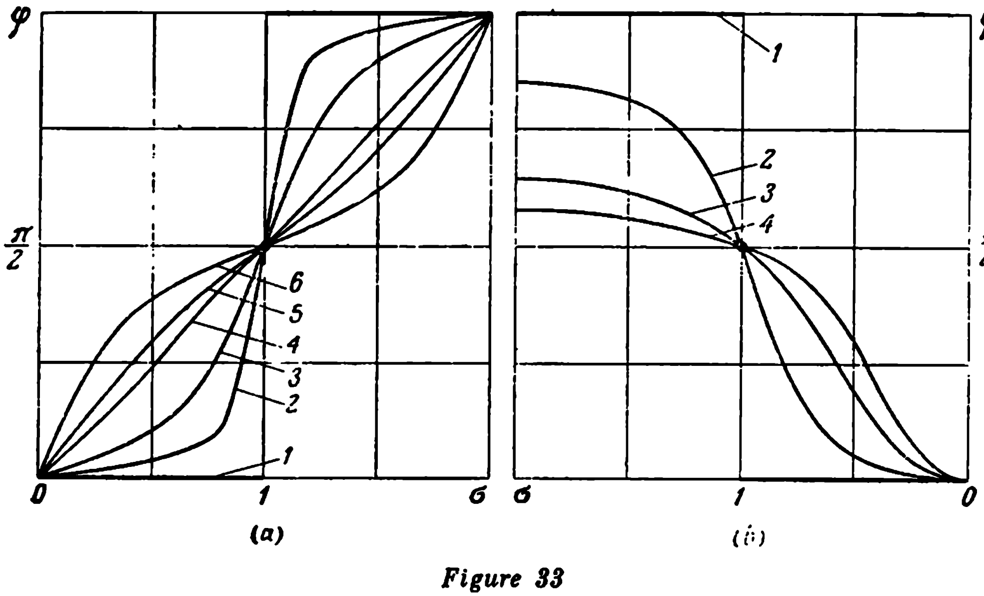


Figure 33

The maxima whose values are $1/2\beta$ are reached at $\sigma = 1$. The difference in the positions of the maxima in Figs. 31*a* and 32*a* is due to the fact that each curve in Fig. 31*a* has been plotted at β = const and in Fig. 32*a*, at $\beta_* =$ const.

Figure 32*b* shows curves of the magnification factor ξ_m versus the argument σ, plotted on the basis of (21), for several values of β.

The phase response curves $\varphi(\sigma)$ constructed on the basis of for-

mula (11) for several β values are presented in Fig. 33*a*. Figure 33*b* shows $\varphi(\sigma)$ curves plotted according to formula (14) for several β_* values. It is important to note that in all cases at $\sigma = 1$ the phase angle of the displacement lags by $\varphi = \pi/2$ behind the phase angle of the force.

On the segment $0 \leqslant \sigma \leqslant 2$ the lag in phase is within the limits $0 \leqslant \varphi \leqslant \pi$.

The amplitude of centrifugally excited forced vibrations is determined by formula (3), Sec. 8. We select as dimensionless variables

$$\tau = \omega_0 t, \quad \xi_c = \frac{m_1 + m_0}{m_0 r} x \tag{26}$$

Formula (3), Sec. 8, now takes the following form:

$$\xi_{ca} = \frac{\gamma^2}{\sqrt{(1-\gamma^2)^2 + 4\beta^2\gamma^2}} \tag{27}$$

Consequently $\xi_{ca} = \ddot{\xi}_a$. The curves are presented in Fig. 31*c*. The amplitude response curves of the dimensionless velocity $\dot{\xi}_{ca}(\sigma) = \gamma\xi_{ca}$ are shown in Fig. 34*a* and that of the dimensionless acceleration $\ddot{\xi}_{ca} = \gamma^2\xi_{ca}$ in Fig. 34*b*.

The differential equations of the absolute and relative motions produced by kinematic excitation can be reduced to the form they have in the case of force-excited vibrations. However, the kinematic excitation provides a wider range of possibilities as to the number of schemata and the variety of response characteristics. For comparison, Fig. 35*a* shows all the possible arrangements of force-excited single-degree-of-freedom systems and those of a kinematically excited single-degree-of-freedom system are given in Fig. 35*b*. Arrangements *4* and *16* are universal systems. The rest are obtained by eliminating one or more of the elements.

Introducing into Eq. (9), Sec. 8, the dimensionless parameters

$$\tau = \omega_0 t, \quad \xi_k = \frac{x}{z_a} \tag{28}$$

we obtain

$$\ddot{\xi}_k + 2\beta\dot{\xi}_k + \xi_k = p\cos(\gamma\tau + \psi) \tag{29}$$

where the dimensionless parameters

$$\beta = \frac{b_1 + b_2}{2m\omega_0}; \quad \gamma = \frac{\omega}{\omega_0} \tag{30}$$

The quantity ω_0 is determined by the second of formulas (10), Sec. 8. Using now two more dimensionless parameters

$$u = \frac{c_2}{c_1 + c_2}, \quad v = \frac{b_2}{b_1 + b_2} \tag{31}$$

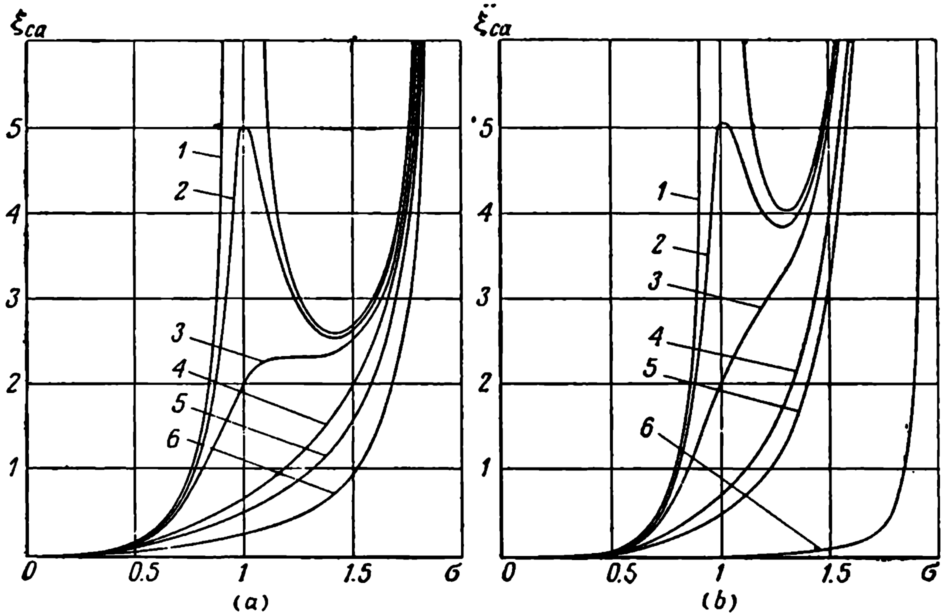


Figure 34

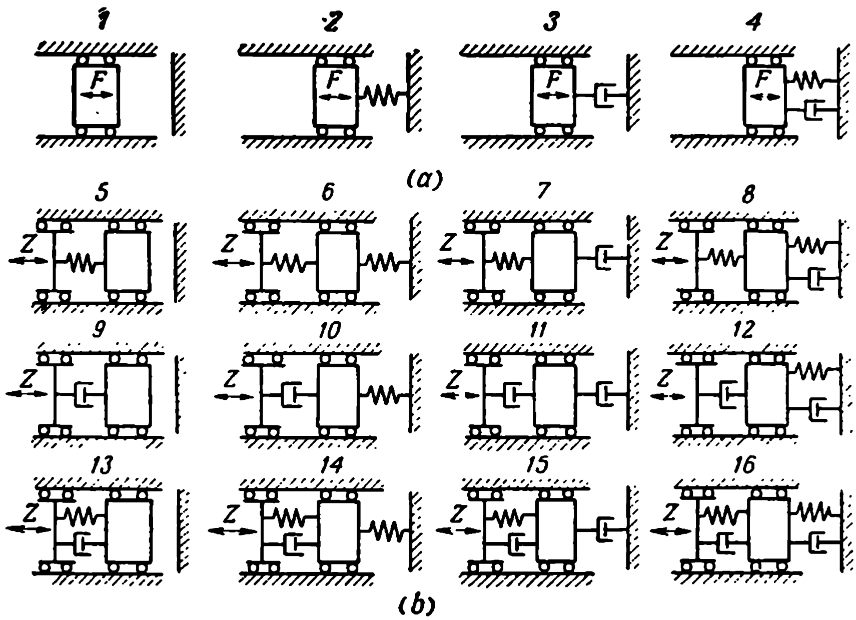


Figure 35

we obtain

$$p = \sqrt{u^2 + 4v^2\beta^2\gamma^2}, \quad \psi = \tan^{-1}\frac{2v\beta\gamma}{u} \tag{32}$$

Thus, in the general case the behaviour of the system is determined by four dimensionless parameters β, γ, u, v, and not by two as with force excitation.

Taking the solution of the form $\xi_k = \xi_{ka}\cos(\gamma\tau - \varphi_{da})$, we can write expressions for the dimensionless absolute displacement amplitude ξ_{ka} and the phase difference angle φ_{da}:

$$\left.\begin{aligned} \xi_{ka} &= \sqrt{\frac{u^2 + 4v^2\beta^2\gamma^2}{(1-\gamma^2)^2 + 4\beta^2\gamma^2}} \\ \varphi_{da} &= \tan^{-1}\frac{2\beta\gamma}{1-\gamma^2} - \tan^{-1}\frac{2v\beta\gamma}{u} \end{aligned}\right\} \tag{33}$$

The quantity ξ_{ka} is the *transmissibility*. It is a characteristic of motion transmission from the controlling element to the controlled one, being equal to the ratio of their displacement amplitudes. With kinematic excitation of vibrations cases can occur, as shown by the second of formulas (33), when the excited vibrations lead in phase the exciting factor. Indeed, the angle φ_{da} may be within the limits $-\pi/2 \leqslant \varphi_{da} \leqslant \pi$, i.e., the lead angle can in the limiting case attain a value of $\pi/2$.

Figure 36 illustrates the amplitude response curves of the transmissibility $\xi_{ka}(\sigma)$ for various combinations of the parameters u and v, each of which takes the values 0; 0.5 and 1.

The corresponding phase response curves $\varphi_{da}(\sigma)$ are shown in Fig. 37, and the amplitude response curves of the dimensionless velocity $\dot{\xi}_{ka}(\sigma) = \gamma\xi_{ka}(\sigma)$ and of the dimensionless acceleration $\ddot{\xi}_{ka}(\sigma) = \gamma^2\xi_{ka}(\sigma)$ are presented in Figs. 38 and 39, respectively. To correlate the arrangements in Fig. 35*b* with the curves, the order numbers of the arrangements to which the curves in Figs. 36 through 43 refer are given in Table 6. The numbers in parentheses refer to arrangements of limiting cases when $\beta = 0$ or $\gamma = 0$.

To plot the response curves for a kinematically excited relative motion we now introduce the following dimensionless variables into Eq. (13), Sec. 8:

$$\tau = \omega_0 t, \quad \eta = \frac{y}{z_a} \tag{34}$$

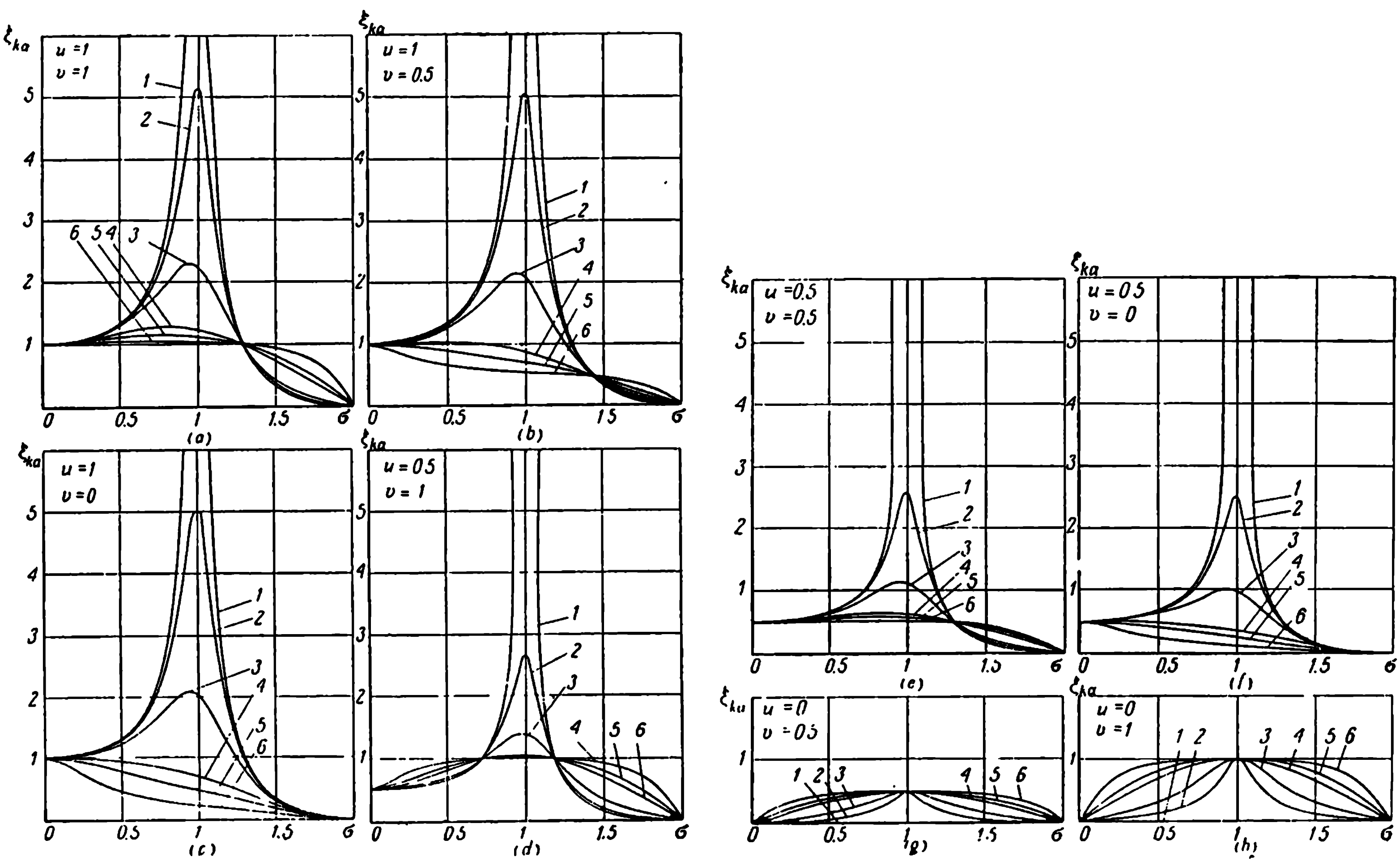
The result takes the form

$$\ddot{\eta} + 2\beta\dot{\eta} + \eta = q\cos(\gamma\tau + \chi) \tag{35}$$

where

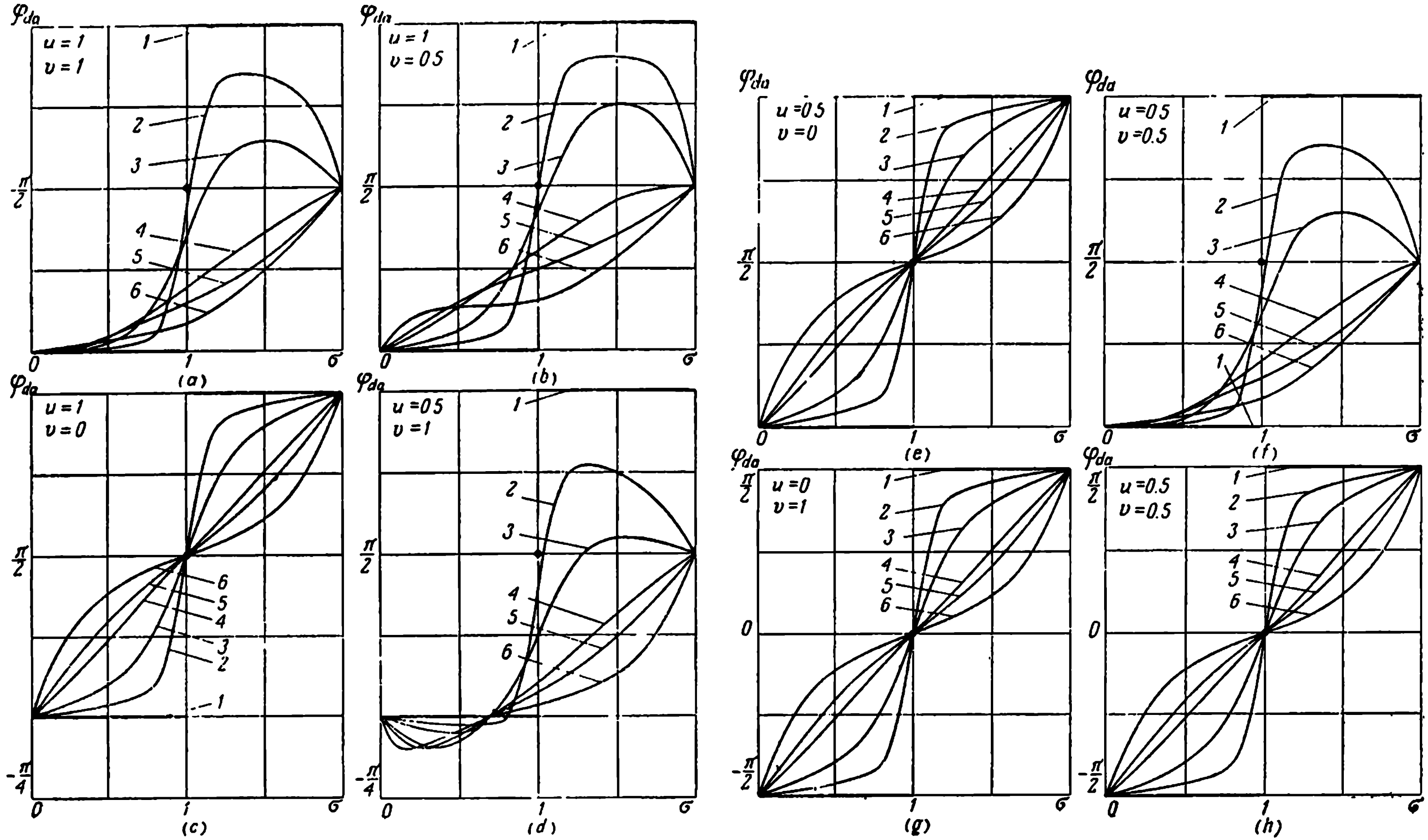
$$q = \sqrt{(1-u-\gamma^2)^2 + 4(1-v)^2\beta^2\gamma^2}$$

$$\chi = \tan^{-1}\frac{2(1-v)\beta\gamma}{1-u-\gamma^2} \tag{36}$$



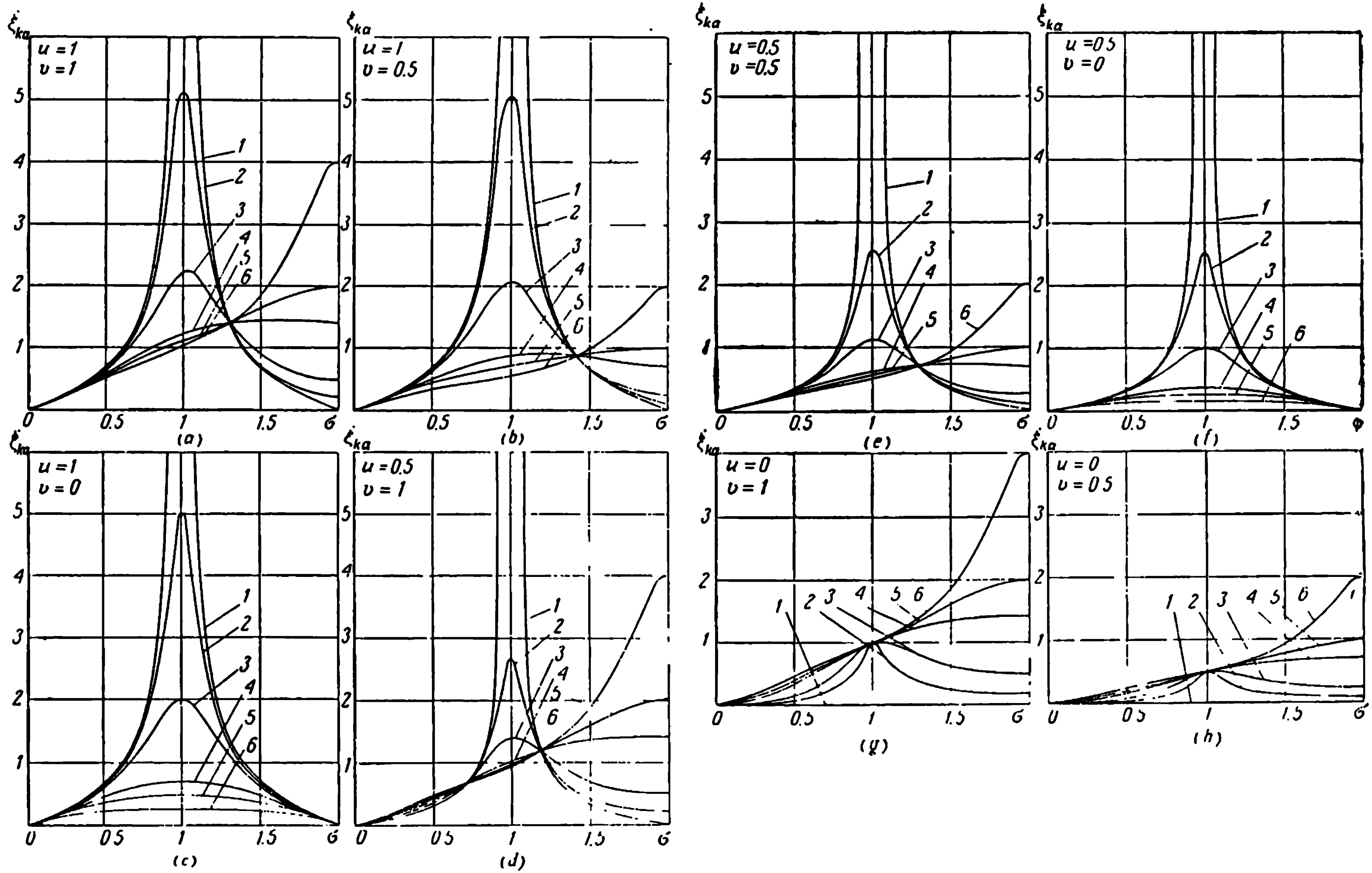
ξ_{ka}
u = 1
v = 1
(a)
u = 1
v = 0.5
(b)
u = 1
v = 0
(c)
u = 0.5
v = 1
(d)
u = 0.5
v = 0.5
(e)
u = 0.5
v = 0
(f)
u = 0
v = 0.5
(g)
u = 0
v = 1
(h)
σ

Figure 36



φda
u = 1
v = 1
(a)
u = 1
v = 0.5
(b)
u = 1
v = 0
(c)
u = 0.5
v = 1
(d)
u = 0.5
v = 0
(e)
u = 0.5
v = 0.5
(f)
u = 0
v = 1
(g)
u = 0.5
v = 0.5
(h)

Figure 37



ξ_{ka}
u = 1
v = 1
(a)
u = 1
v = 0.5
(b)
u = 1
v = 0
(c)
u = 0.5
v = 1
(d)
u = 0.5
v = 0.5
(e)
u = 0.5
v = 0
(f)
u = 0
v = 1
(g)
u = 0
v = 0.5
(h)

Figure 38

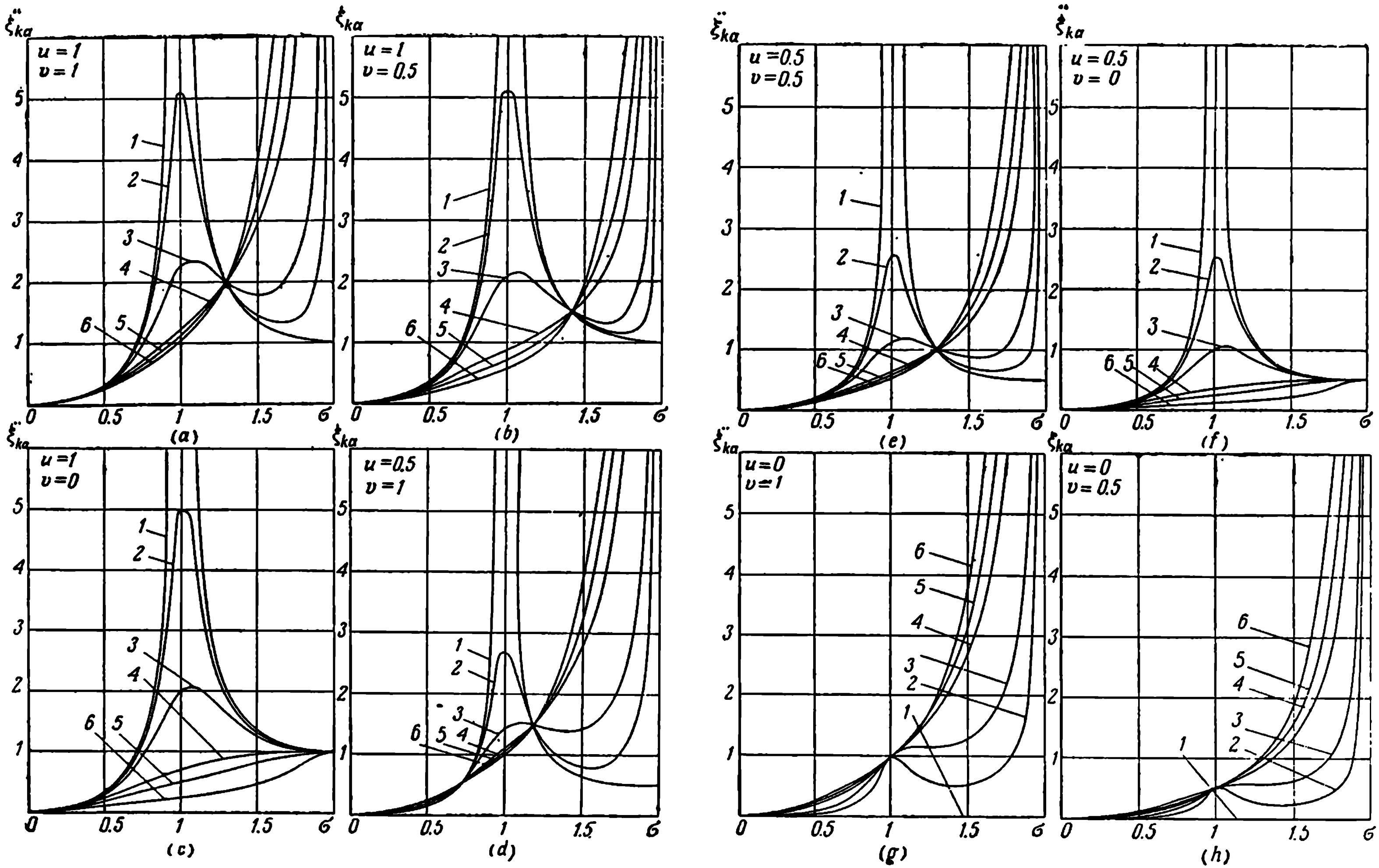


Figure 39

The parameters β, γ, u and v are determined by the relations (30) and (31) and ω_0 by formula (10), Sec. 8.

Assuming the solution to have the form $\eta = \eta_a \cos(\gamma\tau - \varphi_{rel})$ we obtain

$$\left.\begin{aligned}\eta_a &= \sqrt{\frac{(1-u-\gamma^2)^2+4(1-v)^2\beta^2\gamma^2}{(1-\gamma^2)^2+4\beta^2\gamma^2}}\\ \varphi_{rel} &= \tan^{-1}\frac{2\beta\gamma}{1-\gamma^2}-\tan^{-1}\frac{2(1-v)\beta\gamma}{1-u-\gamma^2}\end{aligned}\right\} \tag{37}$$

TABLE 6

Designation of characteristics in Figs. 36-43	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
Order numbers of arrangements in Fig. 35*b*	*13 (5,9)*	*15 (5,11)*	*7 (5)*	*14 (6,9)*	*16 (6,11)*	*8 (6)*	*10 (9)*	*12 (11)*

The response characteristics $\eta_a(\sigma)$, $\varphi_{rel}(\sigma)$, $\dot{\eta}_a(\sigma) = \gamma\eta_a(\sigma)$, $\ddot{\eta}_a(\sigma) = \gamma^2\eta_a(\sigma)$ are shown in Figs. 40 through 43. The correlation of the characteristics with the arrangement numbers in Fig. 35*b* can be seen from Table 6.

In order to present in dimensionless quantities the amplitude response curve of the force acting in the drive of the system with positive motion of its mass element we introduce into formula (18), Sec. 8, the dimensionless amplitude of the force

$$\rho_a = \frac{F_a}{cr} \tag{38}$$

The formula now takes the form

$$\rho_a = \sqrt{(1-\gamma^2)^2+4\beta^2\gamma^2} \tag{39}$$

where β and γ are determined by the expressions (3). Figure 44 shows a family of $\rho_a(\sigma)$ curves.

There are so many varieties of multi-degree-of-freedom coupled systems that it is impossible to give here a brief review of their response characteristics even for the case of two degrees of freedom. We shall therefore limit our discussion to the system presented in Fig. 45. The numbers *1* through *6* have the same meaning as in Fig. 24*a*; the only difference is that a damper *7* with the resistance

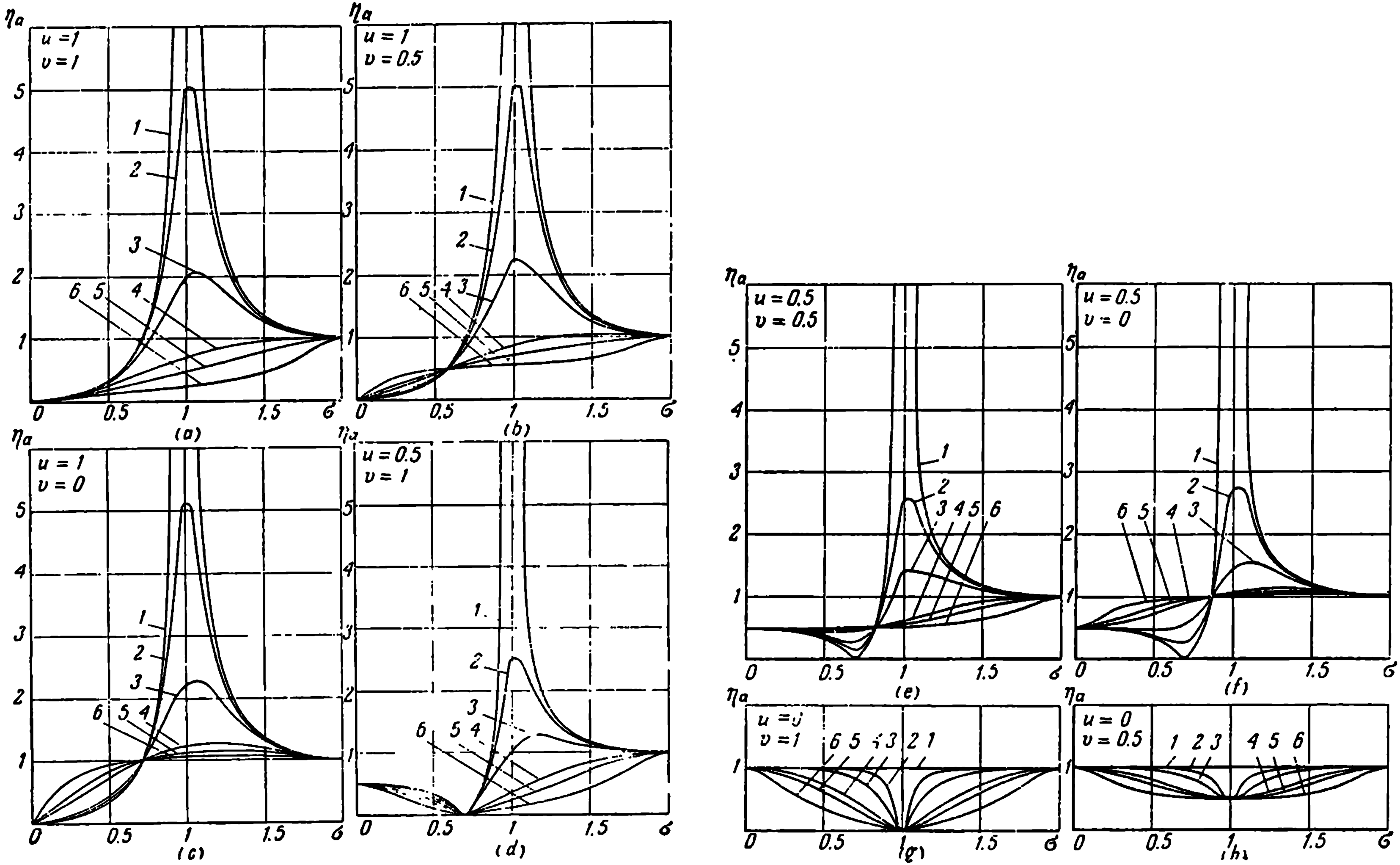


Figure 40

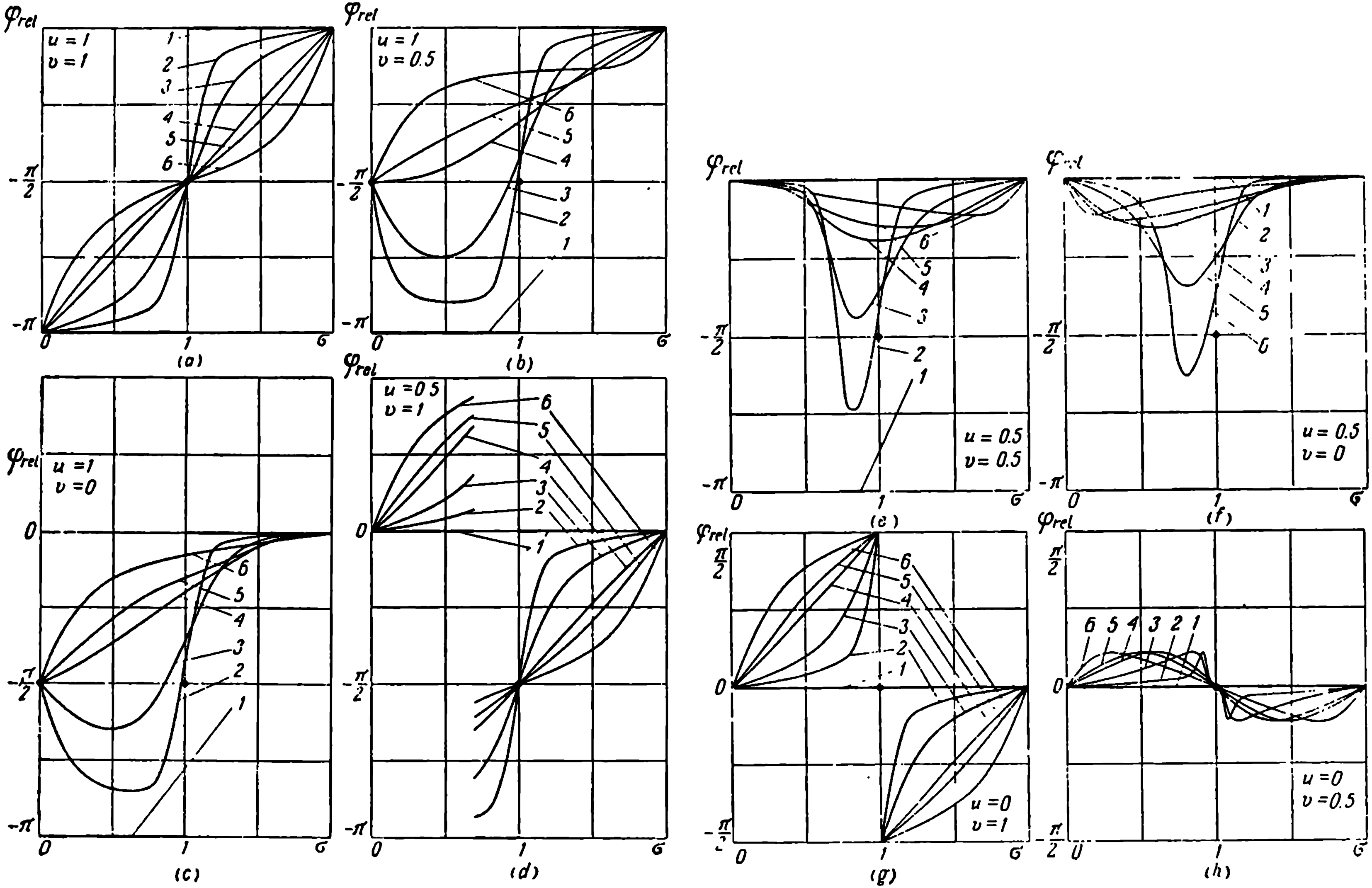


Figure 41

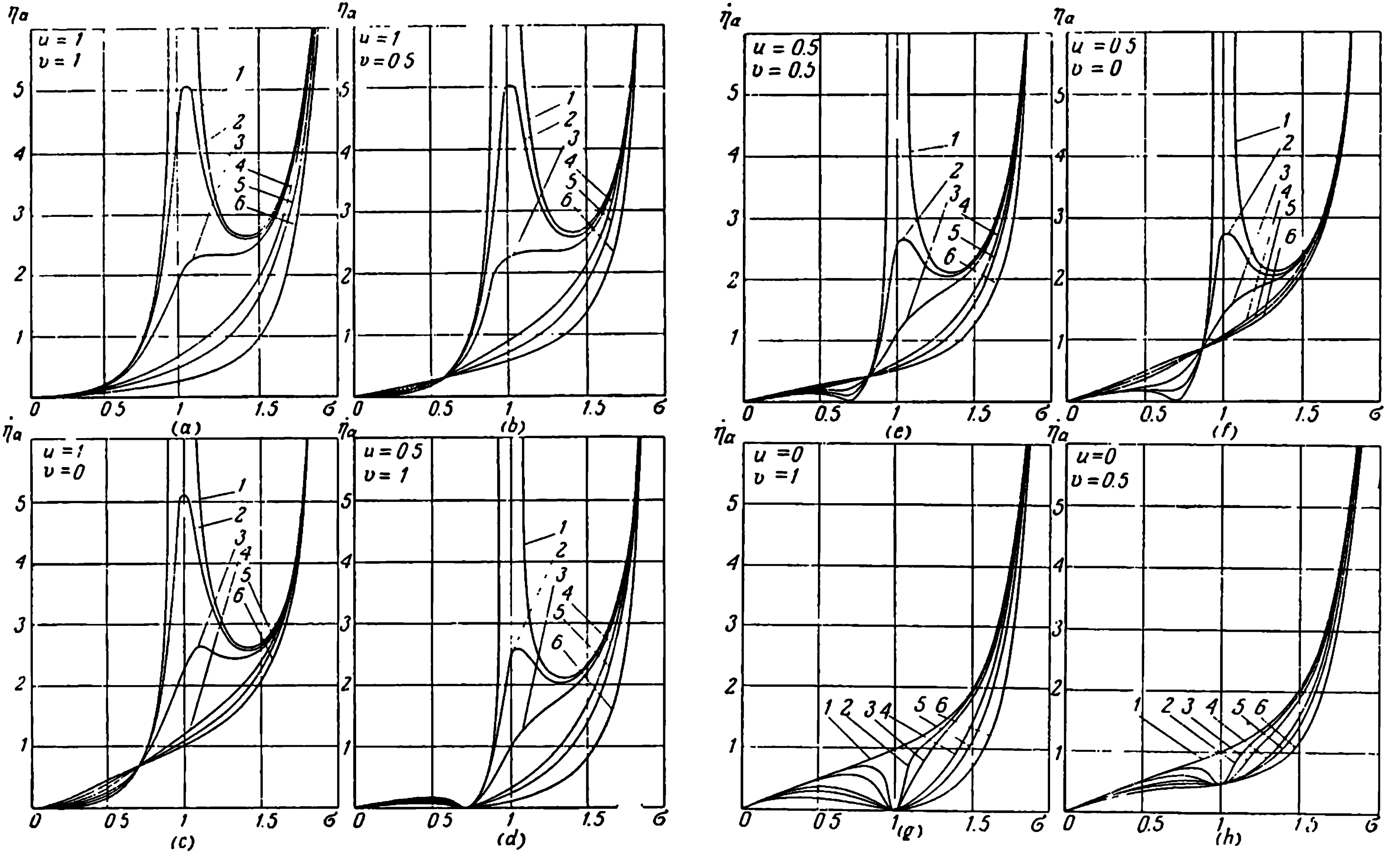


Figure 42

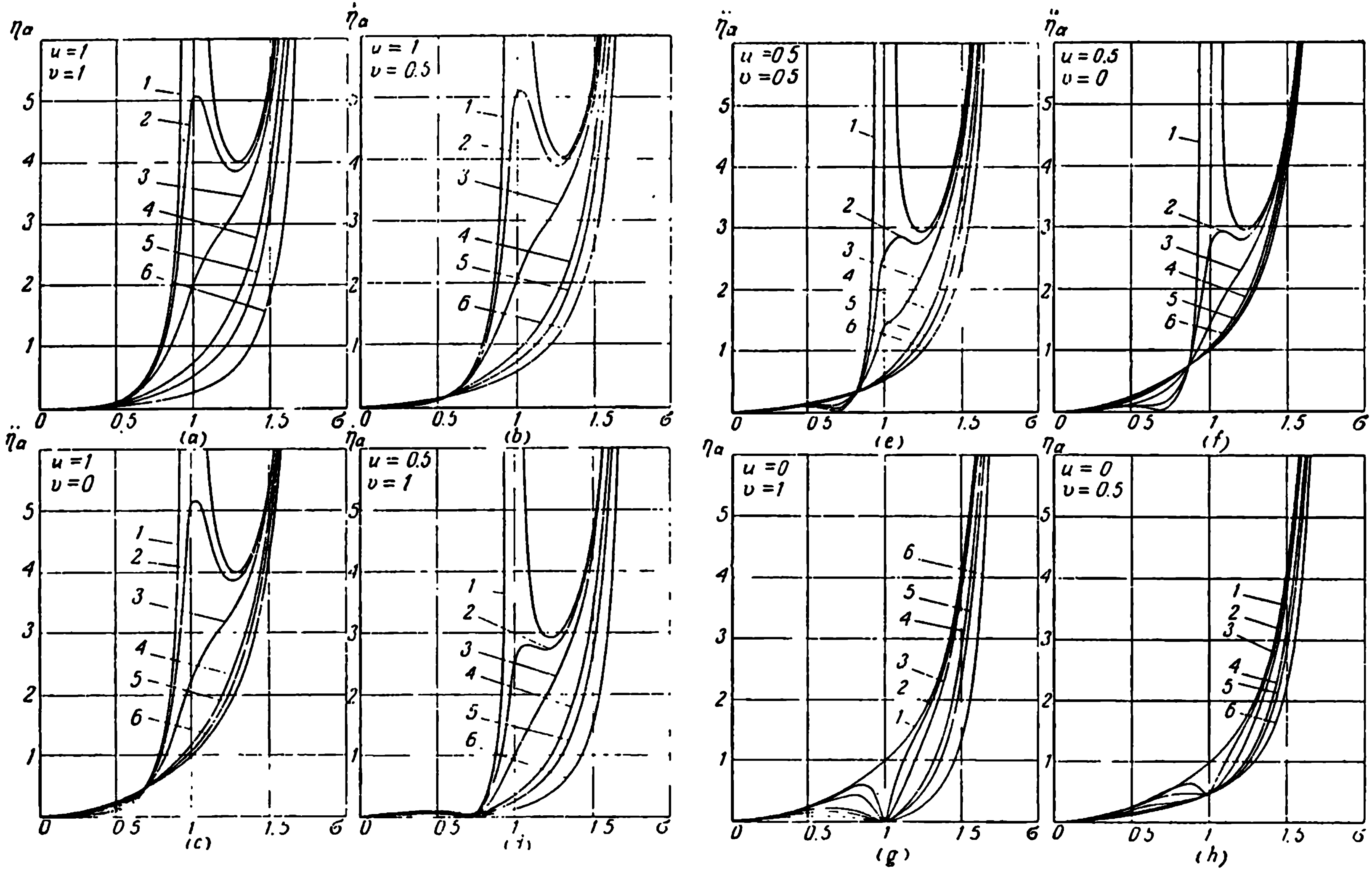


Figure 43

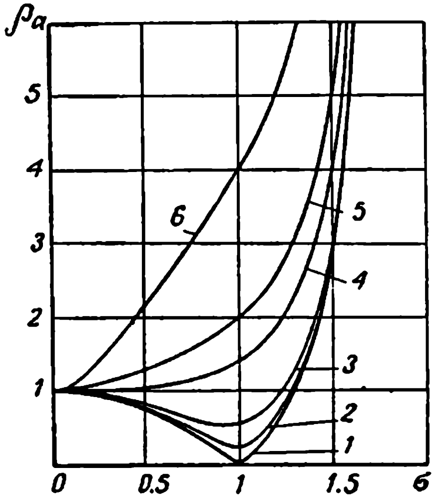


Figure 44

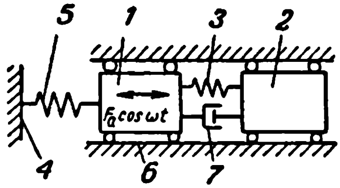


Figure 45

coefficient b has been added. A sinusoidally varying exciting force is applied to element *1*.

With generalized coordinates similar to those used in discussing the diagram in Fig. 24*a* we now set up the equations of motion:

$$\left.\begin{aligned} m_1\ddot{x}_1+b\dot{x}_1-b\dot{x}_2+(c_1+c_2)\,x_1-c_2x_2&=F_a\cos\omega t\\ m_2\ddot{x}_2+b\dot{x}_2-b\dot{x}_1+c_2x_2-c_2x_1&=0\end{aligned}\right\}\tag{40}$$

Introducing the dimensionless variables

$$\xi_1=\frac{c_2}{F_a}x_1,\ \xi_2=\frac{c_2}{F_a}x_2,\qquad \tau=t\sqrt{\frac{c_2}{m_2}}\tag{41}$$

and the parameters

$$\beta=\frac{b}{2\sqrt{m_2c_2}},\ \gamma=\omega\sqrt{\frac{m_2}{c_2}},\ \lambda=\frac{c_1}{c_2},\ \mu=\frac{m_1}{m_2}\tag{42}$$

we now write Eqs. (40) as follows:

$$\left.\begin{aligned} &\mu\ddot{\xi}_1+2\beta\dot{\xi}_1-2\beta\dot{\xi}_2+(\lambda+1)\,\xi_1-\xi_2=\cos\gamma\tau\\ &\ddot{\xi}_2+2\beta\dot{\xi}_2-2\beta\dot{\xi}_1+\xi_2-\xi_1=0\end{aligned}\right\}\tag{43}$$

The particular integral of the simultaneous equations (43) corresponding to stationary forced vibrations can be represented by the expressions

$$\xi_1=\xi_{1a}\cos(\gamma\tau-\varphi_1);\ \xi_2=\xi_{2a}\cos(\gamma\tau-\varphi_2)\tag{44}$$

where

$$\left.\begin{aligned}
\xi_{1a} &= \frac{\sqrt{(AB-\gamma^2C)^2+4\beta^2\gamma^{10}}}{A^2B-2\gamma^2AC+\gamma^4D}\\
\varphi_1 &= \tan^{-1}\frac{2\beta\gamma^5}{AB-\gamma^2C}\\
\xi_{2a} &= \frac{\sqrt{(AC-\gamma^2D)^2+4\beta^2\gamma^6A^2}}{A^2B-2\gamma^2AC+\gamma^4D}\\
\varphi_2 &= \tan^{-1}\frac{2\beta\gamma^3A}{AC-\gamma^2D}\\
A &= \lambda-\mu\gamma^2;\ B=(1-\gamma^2)^2+4\beta^2\gamma^2\\
C &= 1-\gamma^2+4\beta^2\gamma^2;\ D=1+4\beta^2\gamma^2
\end{aligned}\right\} \tag{45}$$

Though in (45) γ is not the ratio of the frequency of the exciting force to the natural frequency of the undamped system, it is convenient in plotting the response curves to use the argument σ introduced by expression (15). It should be held in mind that in the case considered β is not the damping ratio that is defined by formula (43), Sec. 6.

The above relations permit one to establish a number of important properties of two-degree-of-freedom systems, in particular the existence of two resonances and one antiresonance of the ξ_1 coordinate. It is of interest to note that, in the absence of damping, oscillations of ξ_1 and ξ_2 are in phase before antiresonance and in opposite phase after it.

14. Dynamic Control of Vibrations

Three basic problems of the dynamic control of vibrations will be discussed briefly below: magnification, stabilization and suppression of vibrations in open systems, i.e., systems having no feedback. The dynamic magnification is realized by selecting the parameters of a system in such a way that the system operates near the resonance point of the amplitude response curve. This is feasible in systems having one or several degrees of freedom at sufficiently small dissipative resistances.

A serious shortcoming of single-degree-of-freedom systems operating in the resonance zone is the appearance of large forces transmitted to the fixed foundation through the resilient element at large vibration amplitudes of the inertial element. This can be avoided in multi-degree-of-freedom systems. Consider the arrangement in Fig. 24*a*. Let the exciting force $F_{1a}\cos(\omega t+\psi)$ be applied to element *1* and the force $F_{2a}\cos\omega t$ to element *2*. The differential equations of motion will take the form

$$\left.\begin{aligned}
m_1\ddot{x}_1+(c_1+c_2)\,x_1-c_2x_2 &= F_{1a}\cos(\omega t+\psi)\\
m_2\ddot{x}_2+c_2x_2-c_2x_1 &= F_{2a}\cos\omega t
\end{aligned}\right\} \tag{1}$$

We shall now write the solutions of Eqs. (1) corresponding to steady-state forced vibrations for three particular cases:

Case 1:

$$F_{1a}=F_a, \ \psi=0, \ F_{2a}=0$$

$$x_1=\frac{(c_2-m_2\omega^2)\,F_a}{\Delta}\cos\omega t, \ x_2=\frac{c_2F_a}{\Delta}\cos\omega t \tag{2}$$

where

$$\Delta=(c_1+c_2-m_1\omega^2)(c_2-m_2\omega^2)-c_2^2 \tag{3}$$

Case 2:

$$F_{1a}=0, \quad F_{2a}=F_a$$

$$x_1=\frac{c_2F_a}{\Delta}\cos\omega t, \quad x_2=\frac{(c_1+c_2-m_1\omega^2)\,F_a}{\Delta}\cos\omega t \tag{4}$$

Case 3:

$$F_{1a}=F_{2a}=F_a, \quad \psi=\pi$$

$$x_1=\frac{m_2\omega^2F_a}{\Delta}\cos\omega t, \quad x_2=\frac{(c_1-m_1\omega^2)\,F_a}{\Delta}\cos\omega t \tag{5}$$

In all the three cases resonance occurs when $\Delta=0$. The frequency corresponding to the first resonance is $\omega=\Omega_1$ and that corresponding to the second one $\omega=\Omega_2$, where the natural frequencies are determined by expression (25), Sec. 11. To reduce the vibratory load transmitted to the fixed support by spring *5* let us select a very small stiffness for this spring, i.e., assume $c_1 \ll c_2$. We then have

$$\Omega_1\approx 0, \ \Omega_2\approx\sqrt{\frac{c_2(m_1+m_2)}{m_1m_2}} \tag{6}$$

In this way a considerable increase in vibration amplitude is attained with small vibratory forces transmitted to the support.

Of great practical interest is the dynamic stabilization of vibrations. Stabilization means keeping constant the vibration amplitude of an inertial element of a system with more or less considerable changes in its mass (or moment of inertia for rotational vibrations) and parameters of some other fixed elements which are directly connected with the inertial element. The problem of stabilization will be discussed using the same simple examples.

If in the second and third cases to which the solutions (4) and (5) apply, we select a combination of c_2 and m_2 so as to fulfill the equality

$$\omega=\sqrt{\frac{c_2}{m_2}} \tag{7}$$

then the vibrations of element *1* will be independent of its mass m_1 and the stiffness c_1 of spring *5* which connects the element to the fixed support *4*. The displacement amplitude of the element

$$x_{1a}=\frac{F_a}{c_2} \tag{8}$$

will then be equal to the static deformation of spring *3* when acted upon by the constant force F_a. Note that when condition (7) is satis-

fied, the vibration amplitude x_{2a} of element *2* becomes highly sensitive to changes in the parameters c_1 and m_1.

The dynamic suppression of vibrations often used to suppress foundation vibrations and torsional vibrations of rotating shafts is a special case of dynamic stabilization when the purpose is to stabilize the amplitude of one of the inertial elements at zero level. Thus, if condition (7) is satisfied in the first of the cases considered when solution (2) of Eq. (1) is valid, the vibration amplitude x_{1a} of element *1* becomes zero whatever the values of parameters c_1 and m_1. In the third case, as can be seen from solution (5), if $\omega = \sqrt{c_1/m_1}$, the vibrations of element *2* will be eliminated, i.e., $x_{2a} = 0$, whatever the values of c_2 and m_2. It is possible to have $x_{2a} = 0$ in the second case too provided $\omega = \sqrt{(c_1+c_2)/m_1}$, i.e., in this case the vibration elimination cannot be made independent of the parameter c_2 [see (4)].

The dynamic suppression of vibrations corresponds to the antiresonance which was mentioned in Sec. 11 in discussing the structure of the solutions of Eq. (29). It should be noted that complete stabilization or suppression of vibrations is possible only in the absence of energy dissipation in a sub-system that acts as vibration stabilizer or suppressor. This refers, for instance, to the sub-system consisting of elements *2* and *3* upon stabilization or suppression of vibrations of element *1*.

With force-excited vibrations their dynamic stabilization and suppression can be accomplished only in systems having at least two degrees of freedom. With kinematic excitation the stabilization of vibrations (though in a somewhat different sense) can be achieved also in a single-degree-of-freedom system.

In fact, if we assume in expressions (33), Sec. 13, that

$$u = 1 - \gamma^2, \quad v = 1 \tag{9}$$

we obtain

$$\xi_{ka} = 1, \quad \varphi_{da} = 0 \tag{10}$$

this result being independent of the values taken by β and γ. This means that the amplitude and phase of the vibrations of inertial element *1* in Fig. 19 are equal respectively to the amplitude and phase of the vibrations of the exciting element *8*, whatever the values of c_2 and b_2[1] if the conditions $\omega = \sqrt{c_1/m_1}$ and $b_1 = 0$ are satisfied. It follows that the links *6* and *7* between the elements *1* and *8* behave as if they were undeformable. It is easy to see the reason for this behaviour if we recall that in a system with positive vibrations of the mass element, when the above conditions are satisfied, the force amplitude in the driving rigid element is zero, as can be seen from the relations (18), Sec. 8, and (39), Sec. 13.

[1] Except, of course, in the case where c_2 and b_2 are simultaneously equal to zero. This case however cannot occur with kinematically excited vibrations.

CHAPTER 3

PARAMETRIC AND NONLINEAR SYSTEMS

15. Parametric Systems

It was mentioned in Section 5 that the term *parametric*[1] is applied to systems described by linear differential equations with variable coefficients, i.e., with coefficients that depend explicitly on the argument (in the case considered, on time). For instance, the differential equation

$$a(t)\ddot{x}+b(t)\dot{x}+c(t)x=0 \tag{1}$$

describes the motion of a parametric single-degree-of-freedom system. The parameters a, b and c of this system vary independently

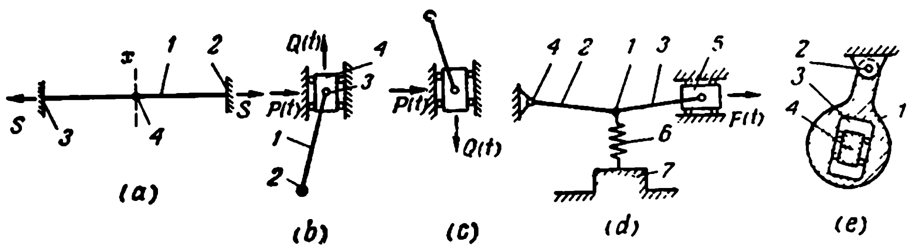


Figure 46

of its motion. Moreover, the motion itself can be excited by varying the parameters. We have discussed above the force and kinematic excitation of motion. We are now concerned with a third excitation method—the *parametric excitation.*

Consider some examples of parametric systems. Figure 46*a* shows a massless string *1* of length $2l$ whose ends are fixed to supports *2* and *3*. A point element *4* of mass m is attached to the string midpoint. The string is tensioned by force S. Denoting by x the displacement from the equilibrium position of element *4* in the plane of the dra-

[1] The term *parametric systems* is sometimes used in a broader sense to denote all systems described by any differential equation with variable parameters, including nonlinear ones.

wing, we can write down for $x \ll l$ the following equation of motion:

$$m\ddot{x}+\frac{S}{l}x=0$$

If the tension force depends on time, for example, $S = S_0-S_1(t)$, where $S_1(t) < S_0$ at any t, then, neglecting energy dissipation, we obtain

$$m\ddot{x}+\frac{1}{l}[S_0-S_1(t)]\,x=0$$

Another example is shown in Fig. 46*b*. The mathematical pendulum consisting of a weightless rod *1* having point mass *2* at its end oscillates about pivot *3*. The pivot is mounted on slide-block *4* which can move vertically under the action of the force $Q(t)$. The pendulum oscillations in a plane give rise to a horizontal reaction $P(t)$. If the length of the pendulum is l, its angle of deviation from the vertical ψ and point mass m, the coordinate of the point mass $x=l\sin\psi$, then the equation of motion takes the form

$$m\ddot{x}=P(t)$$

As the moment of inertia of the system with respect to point *2* is zero, we obtain

$$Q(t)\,l\sin\psi-P(t)\,l\cos\psi=0$$

With small oscillations when $\psi\ll 1$ we can set $\sin\psi=\psi$ and $\cos\psi=1$; then $P(t)=Q(t)\,\psi$ and the equation of motion takes the following form

$$ml\ddot{\psi}-Q(t)\,\psi=0$$

Let the force $Q(t)=-mg-mgk(t)$ where g is the acceleration due to gravity; we now obtain

$$\ddot{\psi}+\frac{g}{l}[1-k(t)]\,\psi=0$$

As is well known, the simple pendulum is in its position of stable equilibrium with the mass in the lower position. The lower position of a parametric pendulum when the function $k(t)$ is periodic, for example sinusoidal, may become unstable with a certain combination of parameters [g/l, the frequency of the function $k(t)$] even at a small amplitude of function $k(t)$. On the contrary, the upper position may become stable and the so-called inverted pendulum shown in Fig. 46*c* can perform periodic oscillations described by the equation

$$\ddot{\psi}-\frac{g}{l}[1+k(t)]\,\psi=0$$

Another parametric system is shown in Fig. 46*d*. Here element *1* of mass m is hinged to two identical massless rods *2* and *3*. The length

of each rod is l. The left end of rod *2* is connected to the fixed pivot *4* and the right end of rod *3* is linked to a pivot mounted on slide-block *5* which can move horizontally in the plane of the drawing. Element *1* is connected to massless spring *6* of stiffness c, the other spring end being connected to fixed stand *7*. In the equilibrium position rods *2* and *3* lie in a straight line. The hinge which connects the rods is at the centre of gravity of element *1*. The force $F(t)$ is applied to slide-block *5*. If the displacements of element *1* from the equilibrium position are small so that $x \ll l$, the equation of motion will become

$$m\ddot{x}+\left[c+\frac{1}{l}F(t)\right]x=0$$

As the last example, consider Fig. 46*e*. The physical pendulum *1* oscillates about the fixed pivot *2*. Slot *3* in the pendulum accommodates a movable mass element *4*. The axis of pivot *2* and the centres of gravity of pendulum *1* and element *4* remain always in one straight line. The motion of element *4* in the slot is determined as a function of time by an external factor. Because of this the moment of inertia J of the system with respect to the axis of pivot *2* and the distance l between the axis of the pivot and the centre of gravity of the system are also functions of time, i.e., $J = J(t)$, $l = l(t)$. The equation of motion can be readily written using the well-known theorem according to which the derivative with respect to time of the moment of momentum relative to a fixed point is equal to the moment of forces about the same point:

$$\frac{d}{dt}\left[J(t)\frac{d\psi}{dt}\right]=-mgl(t)\psi$$

Hence

$$J(t)\ddot{\psi}+\dot{J}(t)\dot{\psi}+mgl(t)\psi=0$$

Let us recall a few important points of the theory of linear differential equations of the second order. Dividing Eq. (1) by $a(t) \neq 0$, we obtain

$$\ddot{x}+p(t)\dot{x}+q(t)x=0 \tag{2}$$

The general solution of the equation

$$x=C_1x_1+C_2x_2 \tag{3}$$

where C_1 and C_2 = arbitrary constants determined from the initial conditions

$x_1(t)$ and $x_2(t)$ = two linearly independent particular solutions.

The linear independence of the solutions means that it is not possible to find such constant numbers that could be used to form the identity $\alpha_1x_1(t) + \alpha_2x_2(t) \equiv 0$. The necessary and sufficient con-

dition of the linear independence of x_1 and x_2 is that the Wronskian of this pair of solutions be different from zero, i.e.,

$$\Delta_W = \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} \neq 0 \tag{4}$$

The Wronskian has the following important property: if it is different from zero at any one value of t, it is not zero at any other t value. On the contrary, if the Wronskian is equal to zero at one value of t, it is identically zero at any t value. It follows that the knowledge of the values of x_1, x_2, $\dot{x}_1$, $\dot{x}_2$ at one moment of time is sufficient to determine whether the solutions are linearly independent.

A pair of linearly independent solutions of x_1 and x_2 is called a fundamental set of solutions since any solution of Eq. (2) may be represented by a linear combination of x_1 and x_2, for example:

$$x_3 = a_{31}x_1(t) + a_{32}x_2(t) \tag{5}$$

where a_{31} and a_{32} are constant coefficients.

In the further treatment only differential equations with periodic coefficients will be considered, i.e., in Eq. (2) it will be assumed that $p(t) = p(t+T)$ and $q(t) = q(t+T)$, where T is the period of variation of the coefficients. Using the notations

$$x_1(t+T) = X_1(t), \quad x_2(t+T) = X_2(t) \tag{6}$$

we may write, on the basis of (5):

$$\left.\begin{aligned} X_1(t) = x_1(t+T) = a_{11}x_1(t) + a_{12}x_2(t) \\ X_2(t) = x_2(t+T) = a_{21}x_1(t) + a_{22}x_2(t) \end{aligned}\right\} \tag{7}$$

The solutions X_1 and X_2 are also a fundamental set since their Wronskian is not zero. Using expressions (7), we may write

$$\Delta_W(t+T) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \Delta_W(t)$$

Since $\Delta_W(t+T) \neq 0$ and $\Delta_W(t) \neq 0$,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \tag{8}$$

Though the coefficients $p(t)$ and $q(t)$ in Eq. (2) are periodic, its solutions $x_1(t)$ and $x_2(t)$ are not necessarily periodic. However, the so-called normal solutions x_n can always be found whose distinctive property is that the following relation is satisfied

$$x_n(t+T) = \lambda x_n(t) \tag{9}$$

where λ is a constant.

With $\lambda = 1$ solution (9) will obviously be periodic, with period T, and when $\lambda = -1$ it will be periodic with period $2T$. From equality (5) we have

$$x_n(t) = c_1 x_1(t) + c_2 x_2(t) \tag{10}$$

whence

$$x_n(t+T) = c_1 x_1(t+T) + c_2 x_2(t+T) \tag{11}$$

Comparing the expressions (9), (10), (11) and (7), we obtain the following relation which holds at any t:

$$[c_1(a_{11}-\lambda) + c_2 a_{21}]\, x_1 + [c_1 a_{12} + c_2(a_{22}-\lambda)]\, x_2 = 0$$

Since x_1 and x_2 are linearly independent, the equalities

$$c_1(a_{11}-\lambda) + c_2 a_{21} = 0, \quad c_1 a_{12} + c_2(a_{22}-\lambda) = 0$$

must hold simultaneously.

Taking into account that $c_1 \neq 0$ and $c_2 \neq 0$, we obtain the so-called characteristic equation:

$$\begin{vmatrix} a_{11}-\lambda & a_{21} \\ a_{12} & a_{22}-\lambda \end{vmatrix} = 0 \tag{12}$$

Its roots are independent of which fundamental set of solutions has been used.

We have now a quadratic equation in λ whose roots are different from zero since its absolute term is not zero, according to (8).

Assuming that the roots of the characteristic equation (12), λ_1 and λ_2, are unequal, we multiply both sides of equality (9) by the exponential factor

$$e^{-\mu_j(t+T)} x_{nj}(t+T) = \lambda_j e^{-\mu_j T} e^{-\mu_j t} x_{nj}(t), \quad (j = 1,\ 2) \tag{13}$$

The function $\varphi_j(t) = e^{-\mu_j t} x_{nj}(t)$ is periodic and has the period T if

$$e^{\mu_j T} = \lambda_j \tag{14}$$

In this case the fundamental set of normal solutions can be expressed in the form

$$x_{n1}(t) = e^{\mu_1 t}\varphi_1(t), \quad x_{n2}(t) = e^{\mu_2 t}\varphi_2(t) \tag{15}$$

If the roots of the characteristic equation (12) are multiple (equal), i. e., $\lambda_1 = \lambda_2 = \lambda$, then the fundamental set of solutions may be written as follows:

$$x_{n1}(t) = e^{\mu t}\varphi(t), \quad x_2 = e^{\mu t}\left[\frac{kt}{T} e^{-\mu T}\varphi(t) + \psi(t)\right] \tag{16}$$

where $e^{\mu T} = \lambda$, $\varphi(t)$ and $\psi(t)$ are periodic functions with the period T, and k is a constant. The solution x_2 is normal only with $k = 0$.

The parameters μ and μ_j, called characteristic indices, generally speaking, are complex quantities.

The solution $x(t)$ is called *stable* if in the course of time up to $t = \infty$ it remains finite. With unequal roots of the characteristic equation, as can be seen from expressions (15), the solutions are stable if the real parts of the characteristic indices are negative or zero. With equal roots of the characteristic equation the solutions are stable, as shown by expressions (16), if the real part of the characteristic is negative. If it is equal to zero, the solutions are stable only with $k = 0$.

The solution is periodic here as with purely imaginary characteristic indices μ_1 and μ_2 in the preceding case if the imaginary part of μ is the product of a rational number and $2\pi/T$. For example, if

$$\operatorname{Im}\mu = \frac{2\pi}{T} \cdot \frac{l}{n}$$

where l and n are relatively prime integers, and T is the period of the functions $q(t)$ and $p(t)$ in Eq. (2), then the period of the solution is equal to nT.

In general the solutions of Eq. (2) cannot be obtained in closed form or expressed in terms of quadratures by using only elementary functions. There are methods of constructing the solutions in the form of infinite series. Special functions are often introduced to present the solutions.

Equations of type (2), for example,

$$\ddot{z} + p(t)\dot{z} + q(t) = 0$$

may be reduced to the form

$$\ddot{x} + f(t)x = 0 \tag{17}$$

by substituting

$$z = xe^{-\frac{1}{2}\int p(t)\,dt} \tag{18}$$

If $f(t)$ is a periodic function, expression (17) is called a Hill's equation. The term is often applied to the special case when $f(t)$ is an even function which can be expanded in a Fourier series in cosine terms.

A special case of Hill's equation has been studied extensively; it is the Mathieu equation:

$$\ddot{x} + (a - 2q\cos 2\omega t)x = 0 \tag{19}$$

If $x'(t)$ is a solution of the Mathieu equation or of a Hill's equation in which $f(t)$ is an even function, then $x'(-t)$ is also a solution. It follows that

$$x_1 = \frac{1}{2}[x'(t) + x'(-t)] \quad \text{and} \quad x_2 = \frac{1}{2}[x'(t) - x'(-t)]$$

are a fundamental set of solutions, where x_1 is an even and x_2 an odd function.

Whether the solutions x_1 and x_2 are stable or unstable depends on the relation between the parameters a and q in Eq. (19). The boundaries between stable and unstable solutions are the periodic solutions known as the Mathieu functions. The notation $ce_n(\omega t, q)$ is used for even Mathieu functions, and $se_n(\omega t, q)$ for odd ones, where $n^2 = a$ with $q = 0$. If n is an integer, we have Mathieu functions of integral order; if n is a fraction, they are Mathieu functions of fractional order.

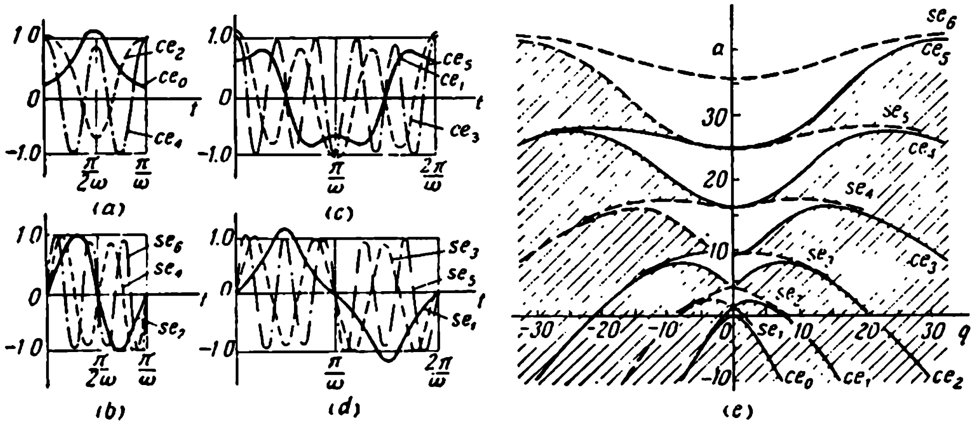


Figure 47

If n is an integer ($n = 0, 1, 2, \ldots$), the functions $ce_{2n}(\omega t, q)$ have the period π/ω and can be expanded in a series in $\cos 2k\omega t$ terms with a constant term depending on q but not on t; the functions $ce_{2n+1}(\omega t, q)$ have the period $2\pi/\omega$ and can be expanded in a series in $\cos k\omega t$ terms without any constant term; the functions $se_{2n+1}(\omega t, q)$ have the period $2\pi/\omega$ and can be expanded in a series in $\sin k\omega t$ terms; the functions $se_{2n+2}(\omega t, q)$ have the period π/ω and can be expanded in series in $\sin 2k\omega t$ terms ($k = 1, 2, \ldots$). In the interval $0 < t \leqslant \pi/\omega$ all these functions have a number of zero values equal to their subscript.

The Mathieu functions of integral order are normalized so that

$$\frac{\omega}{\pi}\int\limits_0^{\frac{2\pi}{\omega}} ce^2(\omega t, q)\,dt = \frac{\omega}{\pi}\int\limits_0^{\frac{2\pi}{\omega}} se^2(\omega t, q)\,dt = 1 \tag{20}$$

The graphs of the functions $ce_0(\omega t, 2)$, $ce_2(\omega t, 2)$ and $ce_4(\omega t, 2)$ are shown in Fig. 47*a*; of the functions $se_2(\omega t, 2)$, $se_4(\omega t, 2)$ and $se_6(\omega t, 2)$ in Fig. 47*b*; of the functions $ce_1(\omega t, 2)$, $ce_3(\omega t, 2)$ and $ce_5(\omega t, 2)$ in Fig. 47*c*; and of the functions $se_1(\omega t, 2)$, $se_3(\omega t, 2)$ and $se_5(\omega t, 2)$ in Fig. 47*d*.

The shaded areas in Fig. 47*e* in the *qa* plane are unstable regions for the solutions of the Mathieu equation. The boundary lines of the regions correspond to Mathieu functions of integral order. The figure shows that the lower the value of $|q|$, the wider is the stable region in the *a*-axis direction. With sufficiently large $|q|$ values the stable regions taper out.

16. Autonomous Nonlinear Systems

Systems whose behaviour is described by nonlinear differential equations are called *nonlinear*. If a differential equation does not contain terms explicitly depending on time, the system described by this equation is called *autonomous* (see Sec. 5). The study of nonlinear systems often proves a difficult problem. There are only a few types of nonlinear differential equations whose exact general solutions can be expressed in closed form by elementary functions.

In studying the solutions of differential equations the analysis of the structure of the state space is very important. The coordinates of the state space of an autonomous system are generalized coordinates and generalized velocities. Hence, the state space of a system having n degrees of freedom is a $2n$-dimensional space. The state space of a single-degree-of-freedom system is two-dimensional and can be graphically represented in a plane (state plane).

Consider the equation of motion (4) of an autonomous conservative linear system which was discussed in Section 6:

$$\ddot{x}+\omega_0^2 x=0$$

Its solution (11), Sec. 6, is

$$x=x_a\cos(\omega_0 t-\varphi)$$

whence

$$v=\dot{x}=-x_a\omega_0\sin(\omega_0 t-\varphi)$$

As mentioned above, the x-, v-plane is called the *state plane*. To each moment of time t there corresponds a pair of x and v values which are the coordinates of a point in the state plane. This point is called the *representative* or *mapping point*. The representative point describes in the course of time a line called the *phase* or *state trajectory*.

The equations $x=x(t)$ and $v=v(t)$ may be regarded as equations of the state trajectory in parametric form. Eliminating the parameter t, we obtain the equation of the state trajectory $f(x, v)=0$. For the case considered we can write:

$$\frac{x}{x_a}=\cos(\omega_0 t-\varphi),\quad \frac{v}{x_a\omega_0}=-\sin(\omega_0 t-\varphi) \tag{1}$$

Squaring equalities (1) and adding them up, we obtain the equation of the state trajectory:

$$\frac{x^2}{x_a^2}+\frac{v^2}{x_a^2\omega_0^2}=1 \tag{2}$$

Since x increases with $v>0$ and decreases with $v<0$, the representative point moves clockwise.

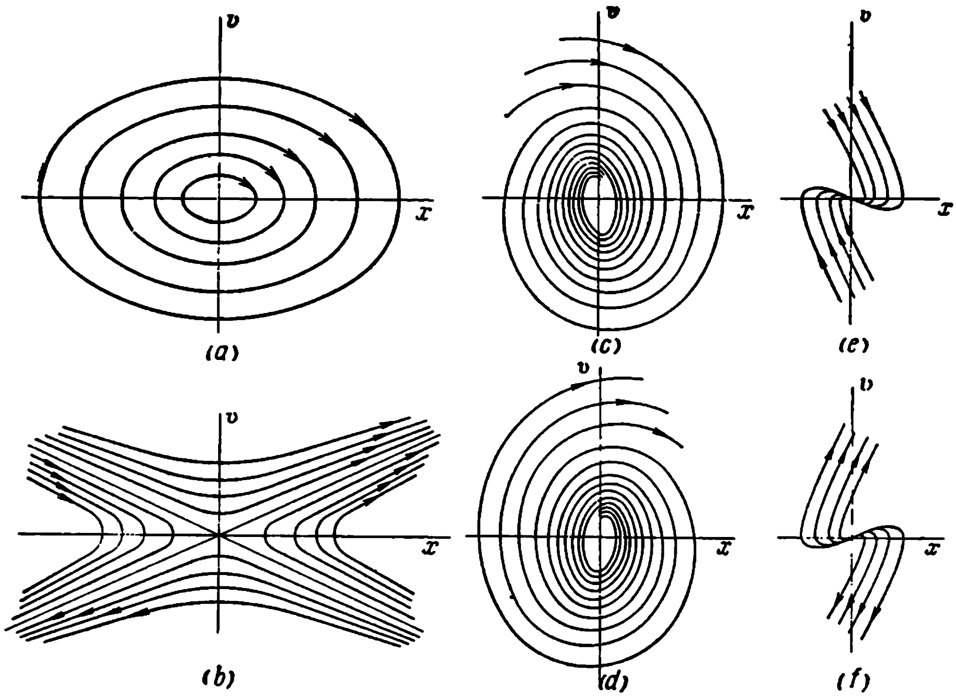


Figure 48

For various values of the amplitude x_a which, according to formula (11), Sec. 6, depends on the initial conditions x_0 and v_0, viz.

$$x_a=\sqrt{x_0^2+\frac{v_0^2}{\omega_0^2}} \tag{3}$$

the state trajectories are ellipses of different size as shown in Fig. 48*a* (the arrows indicate the direction in which the representative points move). Thus, the structure of the state space of the differential equation considered is very simple: the entire state plane is filled by ellipses encircling one another. One and only one state trajectory passes through each point in the state plane. The origin of the coordinates is an exception as no state trajectory passes through it.

The velocity of the representative point is called the *state velocity*. The modulus of the state velocity

$$u=\sqrt{\dot{x}^2+\ddot{x}^2} \tag{4}$$

In our case

$$u = x_a\omega_0 \sqrt{\sin^2(\omega_0 t - \varphi) + \omega_0^2 \cos^2(\omega_0 t - \varphi)} \qquad (5)$$

At no value of $x_a > 0$ can the state velocity be zero, whatever the value of t, since the sine and cosine of the same argument cannot be zero simultaneously. At $x_a = 0$ the representative point is at the origin and its state velocity equals zero. This corresponds to the equilibrium state.

We now replace the second-order differential equation (4), Sec. 6, by a set of two first-order equations:

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\omega_0^2 x \qquad (6)$$

Dividing the second of Eqs. (6) by the first one, we obtain

$$\frac{dv}{dx} = -\omega_0^2 \frac{x}{v} \qquad (7)$$

This equation defines the integral curves which coincide in our case with the state trajectories. The left-hand side of Eq. (7) is the slope of the tangent to the state trajectory. The curve crossing several state trajectories at points of equal slope is called an *isocline*. For this line dv/dx = const. In our case this yields $-\omega_0^2 x/v$ = const or

$$v = kx \qquad (8)$$

where k = const.

Consequently here the straight lines passing through the origin are isoclines.

Integrating Eq. (7), we find that

$$\frac{x^2}{2} + \frac{v^2}{2\omega_0^2} = c$$

Hence, assuming $2c = x_a^2$, we obtain the solution (2).

At the origin we have the indeterminacy $dv/dx = 0/0$. It follows that at the origin there exists no definite direction of the tangent line and the origin is a singular point of the centre type. The term *centre* denotes an isolated singular point which is not passed through by any state trajectory and is encircled by closed state trajectories, one inside the other.

It should be noted that a closed state trajectory corresponds to a periodic oscillation.

The singular point at the origin corresponds to the equilibrium state of the system. Here $\dot{x} = 0$ and $\ddot{x} = 0$. The centre represents the state of stable equilibrium, since the representative point displaced but slightly from the centre will move in its immediate neighbourhood. A pendulum displaced from its state of stable equilibrium behaves in this manner (Fig. 23*d*).

The pendulum motion in the neighbourhood of the position of unstable equilibrium (Fig. 23*e*) can be approximately described by the equation

$$\ddot{x}-\omega_0^2 x=0 \tag{9}$$

where ω_0 must not be regarded as the natural frequency.

Using the same procedure as in the preceding case, we replace Eq. (9) by the following system of equations:

$$\frac{dx}{dt}=v, \quad \frac{dv}{dt}=\omega_0^2 x \tag{10}$$

Hence, dividing the second equation by the first, we obtain

$$\frac{dv}{dx}=\omega_0^2\frac{x}{v} \tag{11}$$

Integration of this equation yields

$$v^2-\omega_0^2 x^2=2c \tag{12}$$

i.e., the equation of a hyperbola whose asymptotes can be determined by setting $c=0$. The state plane in the neighbourhood of the origin is represented in this case by two families of hyperbolas (Fig. 48*b*): the upper and the lower at $c<0$, the right and left ones at $c>0$. At the origin there is now a singular point of the saddle type. Two peculiar state trajectories passing through the saddle point are the asymptotes separating the two families of state trajectories and therefore called *separatrices*.

The differential equation of the free motion of a dissipative system [see (28), Sec. 6]

$$\ddot{x}+2h\dot{x}+\omega_0^2 x=0$$

can be replaced by an equivalent system:

$$\frac{dx}{dt}=v, \quad \frac{dv}{dt}=-(2hv+\omega_0^2 x) \tag{13}$$

whence

$$\frac{dv}{dx}=-\frac{2hv+\omega_0^2 x}{v} \tag{14}$$

The general solution of this differential equation is represented by a family of state trajectories if $0<h<\omega_0$[1] and takes the form

$$(v+hx)^2+(\omega_0^2-h^2)\,x^2=Ce^n \tag{15}$$

where C is the integration constant;

$$n=\frac{2h}{\sqrt{\omega_0^2-h^2}}\tan^{-1}\frac{v+hx}{x\sqrt{\omega_0^2-h^2}} \tag{16}$$

[1] In this case the free motion is decaying vibrations (see Section 6).

It can be proved that expression (15) represents a family of spirals winding onto the origin which in this case is a singular point of the focal type (Fig. 48*c*). In fact, by making the linear substitution

$$y = x\sqrt{\omega_0^2 - h^2}, \quad w = v + hx \tag{17}$$

and introducing polar coordinates

$$y = \rho\cos\varphi, \quad w = \rho\sin\varphi \tag{18}$$

we obtain the equation of a family of logarithmic spirals:

$$\rho = Ce^{\frac{h\varphi}{\sqrt{\omega_0^2 - h^2}}} \tag{19}$$

In the case discussed the focal point represents the state of stable equilibrium. However, if $h < 0$, i.e., we have the so-called *negative damping*, then with $|h| < \omega_0$ the origin is again a singular point of the focal type, but the equilibrium is unstable: the state trajectories unwind out of the focal point (Fig. 48*d*).

If $h > \omega_0$, the corresponding motion being a nonvibratory free motion (see Sec. 6), then the general solution of Eq. (14) is a family of state curves and may be written in the following form:

$$\left[v - \left(h - \sqrt{h^2 - \omega_0^2}\right)x\right]^{-h+\sqrt{h^2-\omega_0^2}} =$$
$$= C\left[v - \left(h + \sqrt{h^2 - \omega_0^2}\right)x\right]^{-h-\sqrt{h^2-\omega_0^2}} \tag{20}$$

This is the equation of a family of curves similar to parabolas (Fig. 48*e*). All the state trajectories enter the origin which in this case is a singular point of the stable node type.

If $h < 0$ and $|h| > \omega_0$, the origin is a singular point of the unstable node type: all the state trajectories issue from it (Fig. 48*f*).

Consider a nonlinear autonomous system having one degree of freedom and described by the equation

$$f(x, v)\frac{dv}{dt} - P(x, v) = 0 \tag{21}$$

Noting that

$$\frac{dv}{dt} = v\frac{dv}{dx}$$

and using the notation

$$Q(x, v) = vf(x, v)$$

we obtain

$$\frac{dv}{dx} = \frac{P(x, v)}{Q(x, v)} \tag{22}$$

where $P(x, v)$ and $Q(x, v)$ are in general nonlinear functions of heir arguments.

If $P(0, 0) = 0$ and $Q(0, 0) = 0$, the origin is a singular point. In this case expansions of P and Q in power series in the neighbourhood of the origin do not contain any constant terms. Let us denote these expansions by the expressions

$$P(x, v) = px + qv + P_2(x, v), \quad Q(x, v) = rx + sv + Q_2(x, v) \tag{23}$$

where P_2 and Q_2 are power series in x and v beginning with terms of the second power at least. With expansions (23) taken into account Eq. (22) takes the form

$$\frac{dv}{dx} = \frac{px + qv + P_2(x, v)}{rx + sv + Q_2(x, v)} \tag{24}$$

This differential equation has at the origin the same singularity (a singular point of the same type) as the linear equation

$$\frac{dv}{dx} = \frac{px + qv}{rx + sv} \tag{25}$$

on the condition that the determinant

$$\Delta = \begin{vmatrix} p & q \\ r & s \end{vmatrix} \neq 0 \tag{26}$$

Table 7 contains the conditions determining the type of singularity at the origin for Eq. (25). The same conditions may be used to determine the type of singularity for Eq. (24), except those specifying a centre: in the case of a nonlinear differential equation the singularity can be either a centre or a focus. To determine which singularity we then have it is necessary to consider the terms of higher degree.

TABLE 7

Singularity type	Principal conditions	Additional conditions	
		Stable	Unstable
Saddle	$(q-r)^2+4ps > 0$; $ps-qr > 0$	—	—
Node	$(q-r)^2+4ps > 0$; $ps-qr < 0$ $(q-r)^2+4ps=0$	$q+r<0$	$q+r>0$
Focus	$(q-r)^2+4ps < 0$; $q+r \neq 0$	$q+r<0$	$q+r>0$
Centre	$(q-r)^2+4ps < 0$; $q+r=0$	—	—

Consider the conservative system described by the differential equation

$$m\ddot{x}+f(x)=0 \tag{27}$$

For this system the differential equation of state trajectories takes the form

$$\frac{dv}{dx}=-\frac{f(x)}{mv} \tag{28}$$

The integral of this equation is

$$\frac{mv^2}{2}+\int_0^x f(z)\,dz=C \tag{29}$$

which expresses the law of conservation of energy.

It follows from Eq. (28) that all the state curves cross the x-axis at right angles (since at $v=0$ we have $dv/dx=\infty$). The state curves are symmetric with respect to the x-axis since in Eq. (29) the v term is of degree 2.

The state trajectory can be plotted as follows. Plot the relation between the potential energy $\Pi=\int_0^x f(z)\,dz$ and x on an auxiliary graph (Fig. 49a). Draw a straight line parallel to the x-axis at a distance C_1 equal to the total energy of the system. The value of x varies in the interval where $\Pi \leqslant C_1$. Calculate the values of v for every x by means of formula (29) and plot the points of the state curve. The point at which the potential energy Π is at a minimum (the state of stable equilibrium) is a singular point of the centre type.

In Fig. 49b a similar construction has been made for the case when $\Pi(x)$ has several extreme values. All the points on the abscissa in the state plane which correspond to minima of the potential energy are centres. Singularities of the saddle type (the states of unstable equilibrium) correspond to maxima of the potential energy. The thick line in Fig. 49b represents the separatrix which separates the regions of qualitatively different motions.

A good illustration of the above general statements is provided by the free motion of the undamped physical pendulum. The differential equation of this motion (17), Sec. 6, takes the form

$$J\ddot{\psi}+mgl\sin\psi=0$$

where the angle ψ is measured from the position of stable equilibrium (the lowest position of the centre of gravity).

Since

$$\ddot{\psi}=\dot{\psi}\frac{d\dot{\psi}}{d\psi},\quad \sin\psi=\psi-\frac{\psi^3}{3!}+\frac{\psi^5}{5!}-\dots$$

we may write down the expression

$$\frac{d\dot{\psi}}{d\psi}=\frac{-\omega_0^2\psi+\frac{\omega_0^2}{3!}\psi^3-\frac{\omega_0^2}{5!}\psi^5+\ldots}{\dot{\psi}}$$

where ω_0 is determined by expression (19), Sec. 6.

The origin in the state plane is a singular point. To determine the type of singularity we refer to Table 7. According to formula (24) we have $p=-\omega_0^2$, $q=0$, $r=0$, $s=1$. Using these values, we obtain $(q-r)^2+4ps=-4\omega_0^2<0$, $q+r=0$. It follows that the singular point is a centre.

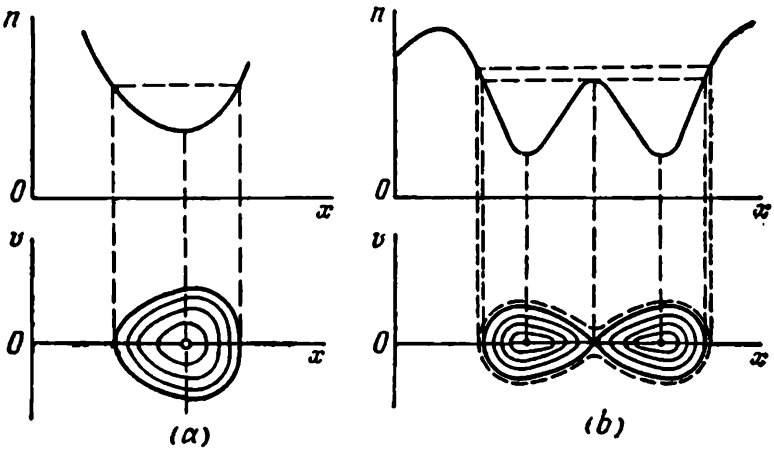


Figure 49

The differential equation of free motion of the same pendulum may now be written, the angle ψ being measured from the position of unstable equilibrium (the uppermost position of the centre of gravity):

$$J\ddot{\psi}-mgl\sin\psi=0$$

whence

$$\frac{d\dot{\psi}}{d\psi}=\frac{\omega_0^2\psi-\frac{\omega_0^2}{3!}\psi^3+\frac{\omega_0^2}{5!}\psi^5-\ldots}{\dot{\psi}}$$

We have now $p=\omega_0^2$, $q=0$, $r=0$, $s=1$ and consequently $(q-r)^2+4ps=4\omega_0^2>0$, $ps-qr>0$. Thus, the origin is a singular point of the saddle type.

The state pattern (a set of state curves) of the system is illustrated in Fig. 50*a*.

If the pendulum swings with linear damping, the differential equation of motion takes the following form (the angle ψ is mea-

sured from the position of stable equilibrium):

$$J\ddot{\psi}+u\dot{\psi}+mgl\sin\psi=0$$

or

$$\ddot{\psi}+2k\dot{\psi}+\omega_0^2\sin\psi=0$$

where

$$k=\frac{u}{2J}<\omega_0$$

Hence

$$\frac{d\dot{\psi}}{d\psi}=\frac{-\omega_0^2\psi-2k\dot{\psi}+\frac{\omega_0^2}{3!}\psi^3-\frac{\omega_0^2}{5!}\psi^5+\ldots}{\dot{\psi}}$$

We have here $p=-\omega_0^2$, $q=-2k$, $r=0$, $s=1$. Hence $(q-r)^2+4ps=4k^2-4\omega_0^2<0$. In this case $q+r=-2k<0$. Using Table 7, we find that the state of stable equilibrium is a singular point of the focal type.

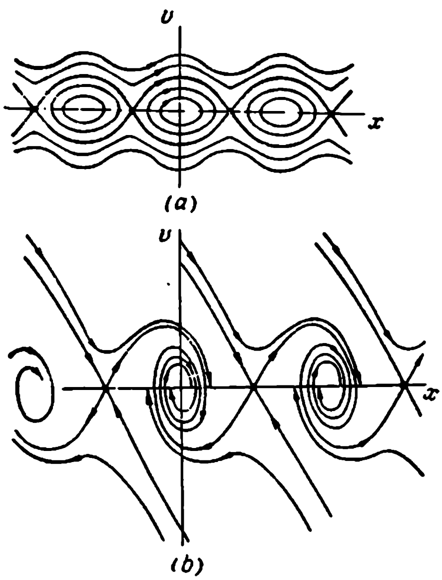


Figure 50

Measuring the angle ψ from the position of unstable equilibrium, we obtain

$$\frac{d\dot{\psi}}{d\psi}=\frac{\omega_0^2\psi-2k\dot{\psi}-\frac{\omega_0^2}{3!}\psi^3+\frac{\omega_0^2}{5!}\psi^5-\ldots}{\dot{\psi}}$$

Here $p=\omega_0^2$; $q=-2k$; $r=0$; $s=1$. Consequently $(q-r)^2+4ps=4k^2+4\omega_0^2>0$, $ps-qr=\omega_0^2>0$. The result shows that the state of unstable equilibrium is a singular point of the saddle type. The state pattern of the system is depicted in Fig. 50*b*.

Consider, as another example, the free vibrations of the system with dry friction schematically shown in Fig. 51*a*. Figure 51*b* shows the characteristic of the friction force. The differential equation of motion about the position where the spring tension is zero takes the form

$$m\ddot{x}+cx=-P\operatorname{sgn}\dot{x} \tag{30}$$

i.e.,

$$\left.\begin{aligned}\ddot{x}+\omega_0^2x&=-\frac{P}{m}\ \text{ at }\ \dot{x}>0\\ \ddot{x}+\omega_0^2x&=\frac{P}{m}\ \text{ at }\ \dot{x}\leqslant 0\end{aligned}\right\} \tag{31}$$

where

$$\omega_0 = \sqrt{\frac{c}{m}}$$

For $\dot{x} \geqslant 0$ we substitute

$$x_1 = x + \frac{P}{m\omega_0^2}$$

and the result obtained is

$$\ddot{x}_1 + \omega_0^2 x_1 = 0$$

This equation, as shown above, gives, on the state plane, a singular point of the centre type at $x_1 = 0$, $\dot{x}_1 = 0$, i.e., at $x = -P/m\omega_0^2$, $\dot{x} = 0$.

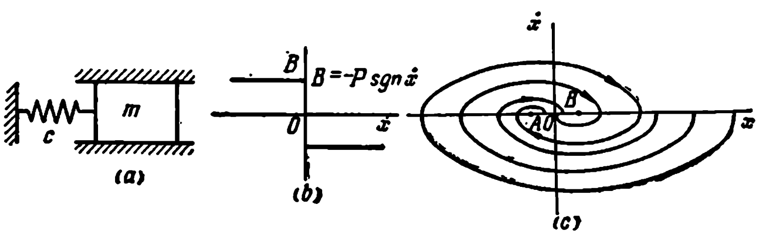


Figure 51

For $\dot{x} \leqslant 0$ the substitution of $x_2 = x - P/m\omega_0^2$ results in the differential equation taking the form

$$\ddot{x}_2 + \omega_0^2 x_2 = 0$$

The state plane coordinates $x_2 = 0$, $\dot{x}_2 = 0$ or $x = P/m\omega_0^2$, $\dot{x} = 0$ determine in this case a singular point of the centre type.

The state pattern of the system is illustrated in Fig. 51*c*. The trajectories on the upper half-plane are ellipses with the centre at point A $(-P/m\omega_0^2, 0)$ and on the lower half-plane—ellipses with the centre at point B $(P/m\omega_0^2, 0)$. When the state trajectory reaches the abscissa segment AB, the motion stops. Segment AB is called accordingly the *stagnation zone* or *critical line*.

The spring force R in the system shown in Fig. 7 was a linear function of the deformation x. The spring characteristic, i.e., the relation $S(x)$, for this case is represented in Fig. 52*a* by the straight line *1*. Two types out of an infinite set of nonlinear characteristics should be noted: the stiffening or hardening spring represented by curve *2* and the softening spring represented by curve *3*. The linear spring has a constant rate (stiffness) $-\partial S/\partial x$. The stiffness of the hardening spring $-\partial S/\partial x$ is an increasing function of the absolute

value of deformation $|x|$. The stiffness of the softening spring $-\partial S/\partial x$ is a decreasing function of $|x|$.

Let $S = -cx - c_1x^3$, where $c > 0$. If $c_1 > 0$, the spring has a hardening characteristic. If $c_1 < 0$, the spring has a softening characteristic. With a hardening spring the differential equation

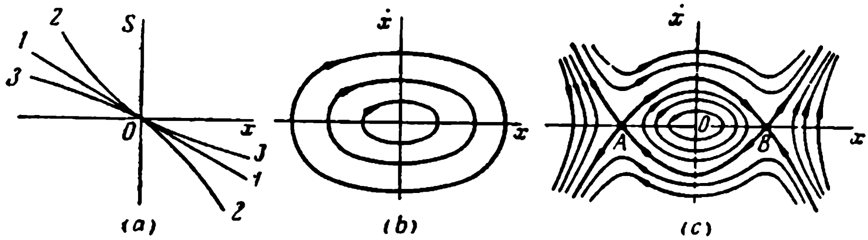


Figure 52

of motion of the system shown in Fig. 7 can be written in the following form:

$$\ddot{x} + \omega_0^2 x + \alpha^2 x^3 = 0$$

where

$$\alpha = \sqrt{\frac{c_1}{m}}; \quad c_1 > 0$$

and x is measured from the equilibrium position.

Hence

$$\frac{d\dot{x}}{dx} = \frac{-\omega_0^2 x - \alpha^2 x^3}{\dot{x}}$$

Here $p = -\omega_0^2$; $q = 0$; $r = 0$; $s = 1$. As shown above, this means that the origin on the state plane is a singular point of the centre type. Figure 52*b* illustrates the state pattern of the system.

For a softening spring the differential equation of motion becomes

$$\ddot{x} + \omega_0^2 x - \alpha^2 x^3 = 0$$

where

$$\alpha = \sqrt{-\frac{c_1}{m}}; \quad c_1 < 0$$

Hence we obtain

$$\frac{d\dot{x}}{dx} = \frac{-\omega_0^2 x + \alpha^2 x^3}{\dot{x}}$$

At the origin we have again a singular point of the centre type. However, if $|x|$ can take sufficiently large values, the system has two positions of unstable equilibrium, apart from having a stable

equilibrium position at $x = 0$. In fact, the equation of equilibrium is a cubic one:

$$\omega_0^2 x - \alpha^2 x^3 = 0$$

It has three real roots: 0, ω_0/α, $-\omega_0/\alpha$. By shifting the reference point of the abscissas to the point ω_0/α we introduce the argument $x_1 = x - \omega_0/\alpha$. Hence

$$\frac{d\dot{x}}{dx} = \frac{2\omega_0^2 x_1 + 3\alpha\omega_0 x_1^2 + \alpha^2 x_1^3}{\dot{x}_1}$$

Since in this case $p = 2\omega_0^2$; $q = 0$; $r = 0$; $s = 1$, the new origin is a singular point of the saddle type. The same is true of the singular point $\left(-\frac{\omega_0}{\alpha}, 0\right)$. The state pattern of the system with a softening spring is illustrated in Fig. 52*c*.

Steady self-excited (self-induced) vibrations may occur in autonomous systems whose motion is described, for instance, by a nonlinear differential equation of the following type:

$$\ddot{x} + \varphi(\dot{x}) + f(x) = 0$$

This is the case when at small values of $|\dot{x}|$ the derivative $d\varphi(\dot{x})/d\dot{x} < 0$ and at large values of $|\dot{x}|$ it is greater than zero, i.e., $d\varphi(\dot{x})/d\dot{x} > 0$.

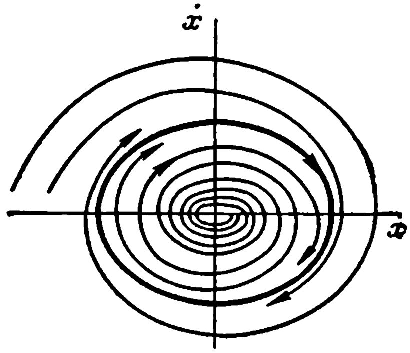


Figure 53

The swing of self-excited vibrations $x_{max} - x_{min}$ attains such a value that the work done by the nonconservative force $\varphi(\dot{x})$ during the period T is zero, i.e.,

$$\int_0^T \varphi(\dot{x})\,\dot{x}\,dt = 0$$

The state pattern of such a system is pictured in Fig. 53. The closed state trajectory represented by the thick line corresponds to steady-state self-excited vibrations and is called the *limit cycle.*

17. Method of Successive Approximations

The idea underlying the method of successive approximations is very simple. It consists in using some suitable iteration process, i.e., repeated application of the same mathematical procedures. The use of the method requires, first of all, a suitable initial appro-

ximation (i.e., an approximate solution), and then a check-up is needed to see whether the process of successive approximations is a convergent one.

Consider the differential equation

$$\ddot{x} + f(x, \dot{x}, t) = 0 \tag{1}$$

where $f(x, \dot{x}, t)$ is a periodic function of the argument t.

We shall seek a periodic solution of Eq. (1) assuming the initial approximation (zero approximation) to be

$$x = x_0(t) \tag{2}$$

This assumption may be based on experience, preliminary qualitative treatment, some plausible reasoning or even on intuition.

Solving Eq. (1) for $\ddot{x}$, we can write

$$\ddot{x} = -f(x, \dot{x}, t) \tag{3}$$

and inserting solution (2) and its derivative into the right-hand side, we have

$$\ddot{x}_1 = -f(x_0, \dot{x}_0, t) \tag{4}$$

The subscript of x is the number of the approximation. Integrating Eq. (4) twice, we obtain

$$x_1 = -\iint f(x_0, \dot{x}_0, t)\, dt^2 \tag{5}$$

The arbitrary constant of the first integration has been taken to be zero as we seek a periodic solution. Taking the arbitrary constant of the second integration to be also zero, we choose in this way as the origin the mean position of the vibrating point. Having determined the approximate value x_1, we seek in the same way the next approximation

$$x_2 = -\iint f(x_1, \dot{x}_1, t)\, dt^2 \tag{6}$$

and so on until we achieve the necessary accuracy.

In practice each successive approximation involves rapidly increasing unwieldiness of derivations and calculation difficulties. One often limits the procedure to one approximation.

For example, let the approximate solution of Duffing's equation,

$$\ddot{x} + \omega_0^2 x + \alpha^2 x^3 = u \cos \omega t \tag{7}$$

be of the form

$$x_0 = a \cos \omega t \tag{8}$$

Substituting it in the differential equation (7), using the trigonometric identity

$$\cos^3\beta \equiv \frac{3}{4}\cos\beta + \frac{1}{4}\cos 3\beta \tag{9}$$

and equating the coefficients of $\cos\omega t$ on the left- and right-hand sides, we obtain the cubic equation

$$(\omega_0^2 - \omega^2)\,a + \frac{3\alpha^2}{4}\,a^3 = u \tag{10}$$

which determines the amplitude a.

Solving now Eq. (7) for the second derivative, introducing into it x_0 and using identity (9), we obtain

$$\ddot{x}_1 = -\omega_0^2 a\cos\omega t - \frac{3\alpha^2}{4}\,a^3\cos\omega t + u\cos\omega t + \frac{1}{4}\,\alpha^2 a^3\cos 3\omega t \tag{11}$$

Substituting into (11) the value of u from expression (10), we obtain

$$\ddot{x}_1 = -\omega^2 a\cos\omega t - \frac{\alpha^2 a^3}{4}\cos 3\omega t \tag{12}$$

and this yields the next approximation

$$x_1 = a\cos\omega t + \frac{\alpha^2 a^3}{36\omega^2}\cos 3\omega t \tag{13}$$

which comprises the first and third harmonics. Further approximations which could enable us to make the amplitude values of the first and third harmonics more precise and to find the amplitudes of the subsequent harmonics are complicated in practice by unwieldy computations and are not as a rule attempted. Tests for the convergence of the iteration process will not be treated here.

One of the versions of the successive approximation methods is Rauscher's method in which a suitable solution of the differential equation describing the free vibrations of a nonlinear system is taken as the initial approximation in solving the differential equation of forced vibrations of the same system. The former equation for a single-degree-of-freedom conservative system can be reduced to a quadrature. Consider the differential equation

$$\ddot{x} + f(x) = u\cos\omega t \tag{14}$$

We begin with the periodic solution of the equation

$$\ddot{x} + f(x) = 0 \tag{15}$$

with frequency ω. We take as the initial conditions

$$x(0) = a,\quad \dot{x}(0) = 0 \tag{16}$$

where the maximum displacement a must be chosen so as to correspond to the angular frequency ω.

The first integral of Eq. (15) with the initial conditions (16) takes the form

$$\frac{\dot{x}^2}{2}=\int_x^a f(\xi)\,d\xi$$

Hence, introducing the notation

$$\Phi(x)=\int_0^x f(\xi)\,d\xi \tag{17}$$

we obtain

$$\dot{x}=-\sqrt{2[\Phi(a)-\Phi(x)]} \tag{18}$$

The second integral can be obtained from Eq. (18) in the form

$$t=t_0(x)=\int_x^a \frac{d\xi}{\sqrt{2[\Phi(a)-\Phi(\xi)]}} \tag{19}$$

For the sake of simplicity we shall assume that the function $f(x)$ in the differential equation (14) is odd, i.e., $f(x)=-f(-x)$. In this case the change of x from a to 0 will take a quarter of the period, i.e., $\pi/2\omega$. Consequently

$$\int_0^a \frac{d\xi}{\sqrt{2[\Phi(a)-\Phi(\xi)]}}=\frac{\pi}{2\omega} \tag{20}$$

Relation (20) determines the value of the maximum displacement a.

Having obtained from expression (19) the initial approximation $t_0(x)$, we insert it into Eq. (14):

$$\ddot{x}+f(x)-u\cos[\omega t_0(x)]=0 \tag{21}$$

Solving this equation in the same way as Eq. (15), we find $t_1(x)$ and, if necessary, proceed to the next approximation and so on.

18. Method of Harmonic Balance

The method of averaging in solving nonlinear differential equations was developed and substantiated in the works of N. Krylov, N. Bogoliubov and Yu. Mitropolsky. The method of harmonic balance is a variety of the method of averaging. The method will be illustrated by the following example. Consider the equation

$$\ddot{x}+f(\dot{x},x)=u\cos\omega t \tag{1}$$

The periodic solution of this differential equation suggests itself in the form of a Fourier series:

$$x=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t) \tag{2}$$

whence

$$\dot{x}=\omega\sum_{n=1}^{\infty}n(-a_n\sin n\omega t+b_n\cos n\omega t) \tag{3}$$

Expanding now the function $f(\dot{x}, x)$ in a Fourier series, we may write:

$$f\Big[\omega\sum_{n=1}^{\infty}n(-a_n\sin n\omega t+b_n\cos n\omega t),$$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\,\omega t)\Big]=$$

$$=\frac{A_0}{2}+\sum_{n=1}^{\infty}(A_n\cos n\omega t+B_n\sin n\omega t) \tag{4}$$

We shall retain only the first harmonics in the expansions and write down the expressions for the coefficients A_0, A_1, B_1 using formulas (2), Sec. 4:

$$A_0=\frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}f\Big(-\omega a_1\sin\omega t+\omega b_1\cos\omega t,$$

$$\frac{a_0}{2}+a_1\cos\omega t+b_1\sin\omega t\Big)\,dt \tag{5}$$

$$A_1=\frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}f\Big(-\omega a_1\sin\omega t+\omega b_1\cos\omega t,$$

$$\frac{a_0}{2}+a_1\cos\omega t+b_1\sin\omega t\Big)\cos\omega t\,dt \tag{6}$$

$$B_1=\frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}f\Big(-\omega a_1\sin\omega t+\omega b_1\cos\omega t,\ \frac{a_0}{2}+$$

$$+a_1\cos\omega t+b_1\sin\omega t\Big)\sin\omega t\,dt \tag{7}$$

From relation (3) we find, limiting the sum to the first harmonic,

$$\ddot{x} = -\omega^2 (a_1 \cos \omega t + b_1 \sin \omega t) \tag{8}$$

Inserting the expressions obtained into the original equation, we have

$$-\omega^2 a_1 \cos \omega t - \omega^2 b_1 \sin \omega t + \frac{A_0}{2} + A_1 \cos \omega t + B_1 \sin \omega t = u \cos \omega t \tag{9}$$

whence

$$\left.\begin{aligned} A_0 &= 0 \\ -\omega^2 b_1 + B_1 &= 0 \\ -\omega^2 a_1 + A_1 &= u \end{aligned}\right\} \tag{10}$$

Replacing the coefficients A_0, A_1, B_1 in expressions (10) by their values from (5), (6), (7), we obtain the equations which determine a_0, a_1 and b_1:

$$\left.\begin{aligned} &\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f\Big(-\omega a_1 \sin \omega t + \omega b_1 \cos \omega t,\ \frac{a_0}{2} + \\ &\qquad + a_1 \cos \omega t + b_1 \sin \omega t\Big)\, dt = 0, \\ &-\omega^2 b_1 + \frac{\omega}{\pi} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f\Big(-\omega a_1 \sin \omega t + \omega b_1 \cos \omega t,\ \frac{a_0}{2} + \\ &\qquad + a_1 \cos \omega t + b_1 \sin \omega t\Big) \sin \omega t\, dt = 0, \\ &-\omega^2 a_1 + \frac{\omega}{\pi} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f\Big(-\omega a_1 \sin \omega t + \omega b_1 \cos \omega t,\ \frac{a_0}{2} + \\ &\qquad + a_1 \cos \omega t + b_1 \sin \omega t\Big) \cos \omega t\, dt = u \end{aligned}\right\} \tag{11}$$

How accurate is the approximate solution obtained depends on how rapidly the series (2) converges.

We take Duffing's equation (7), Sec. 17, as an example, to illustrate the method:

$$\ddot{x} + \omega_0^2 x + \alpha^2 x^3 = u \cos \omega t$$

We take the solution in the form

$$x = \frac{a_0}{2} + a_1 \cos \omega t + b_1 \sin \omega t$$

which can be conveniently transformed as follows:

$$x=\frac{a_0}{2}+c_1\cos(\omega t-\varphi_1)$$

where c_1 and φ_1 are determined in terms of a_1 and b_1 from formulas (4), Sec. 4, where

$$a_1=c_1\cos\varphi_1$$
$$b_1=c_1\sin\varphi_1$$

In the example being considered

$$f(\dot{x},x)=f(x)=\omega_0^2x+\alpha^2x^3$$

Substituting the value of x and retaining only the first harmonic, we obtain the expression

$$f(x)=\frac{\omega^2a_0}{2}+\frac{\alpha^2a_0^3}{8}+\frac{3\alpha^2a_0c_1^2}{4}+\left(\omega_0^2c_1+\frac{3\alpha^2a_0c_1}{4}+\frac{3\alpha^2c_1^3}{4}\right)\cos(\omega t-\varphi_1)$$

Using now the first of equalities (11), we can write

$$a_0\left(\frac{\omega_0^2}{2}+\frac{\alpha^2a_0^2}{8}+\frac{3c_1^2}{4}\right)=0$$

whence $a_0=0$.

The second of the equalities (11) yields

$$\left[(\omega_0^2-\omega^2)c_1+\frac{3}{4}\alpha^2c_1^3\right]\sin\varphi_1=0$$

whence $\varphi_1=0$.

Finally, we find from the third of the equalities (11)

$$c_1(\omega_0^2-\omega^2)+\frac{3}{4}\alpha^2c_1^3=u$$

which coincides with expression (10), Sec. 17.

Using the above approximate solution, one can replace the original nonlinear differential equation by a linear one whose solution coincides with the approximate solution of the original nonlinear equation. Such a transformation of a differential equation is called *equivalent linearization.*

We now replace Duffing's equation discussed above by the linear equation

$$\ddot{x}+k\omega_0^2x=u\cos\omega t \tag{12}$$

Inserting into Eq. (12) the approximate solution of Duffing's equation obtained by the method of harmonic balance, we can write down the following relation:

$$(k\omega_0^2-\omega^2)c_1=u \tag{13}$$

On the other hand, we have

$$(\omega_0^2-\omega^2)\,c_1+\frac{3}{4}\,\alpha^2c_1^3=u \tag{14}$$

Comparing the last two expressions, we obtain

$$k=1+\frac{3\alpha^2c_1^2}{4\omega_0^2} \tag{15}$$

Thus the linearized differential equation (12) becomes

$$\ddot{x}+\left(1+\frac{3\alpha^2c_1^2}{4\omega_0^2}\right)\omega_0^2x=u\cos\omega t \tag{16}$$

The differential equation furnished by equivalent linearization has one important property: the coefficient of x on the left-hand side of the equation, which is the natural frequency squared, depends on the amplitude of vibrations c_1.

19. Small-Parameter Method

The small-parameter method consists in expanding the solution of the differential equation in a power series in a small parameter in the neighbourhood of the so-called generating solution. This is the solution of the generating (abridged) differential equation to which the original equation is reduced if the small parameter vanishes. This method was proposed and substantiated by Henri Poincaré. We shall limit ourselves to the consideration of the techniques of application of one variant of the small-parameter method to the solution of the following differential equation:

$$m\frac{d^2x}{dt^2}+cx+c_1x^3=0 \tag{1}$$

or

$$\frac{d^2x}{dt^2}+\omega_0^2x+\alpha^2x^3=0 \tag{2}$$

where

$$\omega_0=\sqrt{\frac{c}{m}};\quad \alpha^2=\frac{c_1}{m} \tag{3}$$

The solution will be constructed in the form of a series in the neighbourhood of the solution of the following generating equation

$$\frac{d^2x}{dt^2}+\omega_0^2x=0 \tag{4}$$

The concept of the small parameter can be given a precise meaning if the parameter is dimensionless, i.e., if its magnitude is independent of the unit of measurement used. It is therefore reasonable to reduce the differential equation to the dimensionless form

which can be done, for example, by introducing dimensionless variables

$$\tau = \lambda\omega_0 t = \omega t, \;\; \xi = \frac{x}{a} \tag{5}$$

where a = half-swing of vibration displacement
ω = vibration frequency
λ = ratio of vibration frequency ω of the given system to the frequency ω_0 of the generating system.

Introducing the new variables (5) into Eq. (2), denoting differentiation with respect to τ by dots over ξ and using the small parameter

$$\mu = \frac{\alpha^2 a^2}{\omega_0^2} \tag{6}$$

we obtain

$$\lambda^2\ddot{\xi} + \xi + \mu\xi^3 = 0 \tag{7}$$

We now seek the solution of Eq. (7) in the form of the series

$$\xi(\tau) = \xi^{(0)}(\tau) + \mu\xi^{(1)}(\tau) + \mu^2\xi^{(2)}(\tau) + \dots \tag{8}$$

The ratio of the frequencies may also be represented in the form of a series

$$\lambda = 1 + \mu\lambda_1 + \mu^2\lambda_2 + \dots \tag{9}$$

In each particular case the chosen parameter μ is considered small if the power series in μ is convergent. If no suitable parameter is contained in the original differential equation, such a parameter may sometimes be introduced artificially.

We now insert the series (8) and (9) into the original equation (7) (retaining terms of smallness order up to the second, inclusive, i.e., containing μ^2 as a factor):

$$\begin{aligned}[1 + 2\mu\lambda_1 + \mu^2(\lambda_1 + 2\lambda_2) + \dots](\ddot{\xi}^{(0)} + \mu\ddot{\xi}^{(1)} + \mu^2\ddot{\xi}^{(2)} + \dots) +\\ + \xi^{(0)} + \mu\xi^{(1)} + \mu^2\xi^{(2)} + \dots + \mu\xi^{(0)3} + 3\mu^2\xi^{(0)2}\xi^{(1)} + \dots \equiv 0\end{aligned} \tag{10}$$

This substitution transforms the differential equation (7) into identity (10) which holds for any value of μ within the convergency of the series. Coefficients of all powers of μ, beginning with zero power, must therefore be zero. Equating the coefficients to zero, we obtain a sequence of linear differential equations of the second order in the functions $\xi^{(n)}$, ($n = 0, 1, 2, \dots$). The equations can be written as follows:

$$\ddot{\xi}^{(0)} + \xi^{(0)} = 0 \tag{11}$$

$$\ddot{\xi}^{(1)} + \xi^{(1)} = -2\lambda_1\ddot{\xi}^{(0)} - \xi^{(0)3} \tag{12}$$

$$\ddot{\xi}^{(2)} + \xi^{(2)} = -(\lambda_1^2 + 2\lambda_2)\ddot{\xi}^{(0)} - 2\lambda_1\ddot{\xi}^{(1)} - 3\xi^{(0)2}\xi^{(1)} \tag{13}$$

. . . .

In solving the equations we shall require that all the $\xi^{(n)}$ functions have the common period 2π, i.e.,

$$\xi^{(n)}(\tau)=\xi^{(n)}(\tau+2\pi), \quad (n=0, 1, 2, \ldots) \tag{14}$$

We take also the displacement of the system from the equilibrium position at the initial value of the argument to be maximum, i.e., at $\tau=0$

$$\xi=1, \quad \dot{\xi}=0 \tag{15}$$

These initial conditions are distributed between the component functions $\xi^{(n)}$ and their derivatives $\dot{\xi}^{(n)}$ as follows: at $\tau=0$

$$\xi^{(0)}=1, \quad \dot{\xi}^{(0)}=0, \quad \xi^{(k)}=0, \quad \dot{\xi}^{k}=0, \quad (k=1, 2, \ldots) \tag{16}$$

The general solution of the generating equation (11) can be written in the form

$$\xi^{(0)}=a_0\cos\tau+b_0\sin\tau \tag{17}$$

Under the conditions (16) $a_0=1$, $b_0=0$ and consequently

$$\xi^{(0)}=\cos\tau \tag{18}$$

Substituting this solution into the right-hand side of the differential equation (12) and using identity (9), Sec. 17, we obtain

$$\ddot{\xi}^{(1)}+\xi^{(1)}=\left(2\lambda_1-\frac{3}{4}\right)\cos\tau-\frac{1}{4}\cos 3\tau \tag{19}$$

To satisfy the periodicity condition (14) it is necessary to have no resonance term on the right-hand side of Eq. (19), i.e., the coefficient of $\cos\tau$ must be zero:

$$2\lambda_1-\frac{3}{4}=0$$

whence

$$\lambda_1=\frac{3}{8} \tag{20}$$

Using this result, the general solution of Eq. (19) assumes the form

$$\xi^{(1)}=a_1\cos\tau+b_1\sin\tau+\frac{1}{32}\cos 3\tau$$

From the initial conditions (16) we find

$$a_1=-\frac{1}{32}, \quad b_1=0$$

Hence

$$\xi^{(1)}=-\frac{1}{32}\cos\tau+\frac{1}{32}\cos 3\tau \tag{21}$$

We now insert the results (18), (20) and (21) into the right-hand side of the differential equation (13) and after using trigonometric identities we transform it into

$$\ddot{\xi}^{(2)}+\xi^{(2)}=\left(2\lambda_2+\frac{21}{128}\right)\cos\tau+\frac{3}{16}\cos 3\tau-\frac{3}{128}\cos 5\tau \qquad (22)$$

The periodicity condition (14) furnishes the expression

$$2\lambda_2+\frac{21}{128}=0$$

whence

$$\lambda_2=-\frac{21}{256} \qquad (23)$$

Using this result, we can write the general solution of Eq. (22) as follows:

$$\xi^{(2)}=a_2\cos\tau+b_2\sin\tau-\frac{3}{128}\cos 3\tau+\frac{1}{1024}\cos 5\tau$$

From the initial conditions (16) we derive

$$a_2=\frac{23}{1024}, \quad b_2=0$$

Hence

$$\xi^{(2)}=\frac{23}{1024}\cos\tau-\frac{3}{128}\cos 3\tau+\frac{1}{1024}\cos 5\tau \qquad (24)$$

Inserting now the results into the series (8) and (9), we obtain the required solution of the nonlinear differential equation (7) accurate to the terms of second order smallness inclusive

$$\xi=\left(1-\frac{1}{32}\mu+\frac{23}{1024}\mu^2\right)\cos\tau+\frac{1}{32}\mu\left(1-\frac{3}{4}\mu\right)\cos 3\tau+$$
$$+\frac{1}{1024}\mu^2\cos 5\tau \qquad (25)$$

and the frequency ratio

$$\lambda=1+\frac{3}{8}\mu-\frac{21}{256}\mu^2 \qquad (26)$$

Returning now to the original differential equation (2), we write down its solution:

$$x=\left(1-\frac{\alpha^2a^2}{32\omega_0^2}+\frac{23\alpha^4a^4}{1024\omega_0^4}\right)a\cos\omega t+$$
$$+\frac{\alpha^2a^2}{32\omega_0^2}\left(1-\frac{3\alpha^2a^2}{4\omega_0^2}\right)a\cos 3\omega t+\frac{\alpha^4a^5}{1024\omega_0^4}\cos 5\omega t \qquad (27)$$

where

$$\omega=\omega_0\left(1+\frac{3\alpha^2a^2}{8\omega_0^2}-\frac{21\alpha^4a^4}{256\omega_0^4}\right) \qquad (28)$$

Other examples of the application of the small-parameter method will be found in Sections 36 and 42.

20. Method of Fitting

Approximate analytical methods of investigating nonlinear systems have been discussed in Sections 17 through 19. In the present section we shall concern ourselves with the precision method of fitting (jointing, sewing) usually employed in investigating piecewise linear systems, i.e., nonlinear systems whose behaviour within separate sections is described by linear equations. A solution is obtained for each section and the initial conditions for the next section are then made to fit the terminal conditions of the preceding section.

We now turn to the computation procedure using, as an example, the free vibrations of a single-degree-of-freedom system with dry friction (Fig. 51). The differential equation of motion of this system [Eq. (30), Sec. 16] is broken up into two linear equations, viz.:

$$\ddot{x}_1 + \omega_0^2 x_1 = -\frac{P}{m} \quad \text{at} \quad \dot{x} \geqslant 0 \tag{1}$$

$$\ddot{x}_2 + \omega_0^2 x_2 = \frac{P}{m} \quad \text{at} \quad \dot{x} \leqslant 0 \tag{2}$$

The motion in the first stage starts at the initial conditions

$$t = 0, \; x_1 = -a_0, \; \dot{x}_1 = 0 \tag{3}$$

The corresponding general solution of Eq. (1) may be taken in the form

$$x_1 = -\left(a_0 - \frac{P}{m\omega_0^2}\right)\cos\omega_0 t - \frac{P}{m\omega_0^2} \tag{4}$$

Hence the velocity

$$\dot{x} = \left(a_0\omega_0 - \frac{P}{m\omega_0}\right)\sin\omega_0 t \tag{5}$$

The first stage of the motion is completed at $\dot{x}_1 = 0$ at the moment

$$t_1 = \frac{\pi}{\omega_0} \tag{6}$$

Substituting this value of t_1 into Eq. (4), we obtain

$$x_1\left(\frac{\pi}{\omega_0}\right) = a_0 - \frac{2P}{m\omega_0^2} \tag{7}$$

Thus the terminal conditions of the first stage at $t = \pi/\omega_0$ are

$$x_1 = a_0 - \frac{2P}{m\omega_0^2}, \quad \dot{x}_1 = 0 \tag{8}$$

These conditions will be the initial ones for the second stage in which Eq. (2) becomes valid, i.e., at $t=\pi/\omega_0$

$$x_2 = a_0 - \frac{2P}{m\omega_0^2}, \quad \dot{x}_2 = 0 \tag{9}$$

The solution of Eq. (2) corresponding to the initial conditions (9) takes the form

$$x_2 = -\left(a_0 - \frac{3P}{m\omega_0^2}\right)\cos\omega_0 t + \frac{P}{m\omega_0^2} \tag{10}$$

Hence the velocity

$$\dot{x}_2 = \left(a_0\omega_0 - \frac{3P}{m\omega_0}\right)\sin\omega_0 t \tag{11}$$

The second stage of the motion is completed at $\dot{x}_2 = 0$, i.e., at the moment $t = 2\pi/\omega_0$.

It follows that the terminal conditions for the second stage at $t = 2\pi/\omega_0$ will be:

$$x_2 = -\left(a_0 - \frac{4P}{m\omega_0^2}\right), \quad \dot{x}_2 = 0 \tag{12}$$

These conditions will be the initial ones for the third stage for which Eq. (1) again will be valid.

The motion is decaying vibrations where the displacement maxima decrease in an arithmetic progression with the difference $4P/m\omega_0^2$.

When at the end of one of the stages in the sequence the absolute displacement value becomes less than or equal to $P/m\omega_0^2$, the motion will stop.

The possibilities of the method of fitting may be presented more vividly by using, as an example, a system with periodic vibrations. Let a sinusoidal exciting force be applied to the vibrating body of the system discussed above. The differential equation of the vibrations will now take the form

$$m\frac{d^2x}{dt^2} + cx = -P\,\mathrm{sgn}\left(\frac{dx}{dt}\right) + F_a\cos(\omega t + \varphi) \tag{13}$$

where φ is the phase of the exciting force at the starting moment of the first stage; the phase is to be determined.

We now pass to the dimensionless variables

$$\tau = \omega t, \quad \xi = \frac{m\omega^2}{F_a}x \tag{14}$$

and introduce the dimensionless parameters

$$\gamma_* = \frac{1}{\omega}\sqrt{\frac{c}{m}}, \quad p = \frac{P}{F_a} \tag{15}$$

Equation (13) can then be written in the form

$$\ddot{\xi}+\gamma_*^2\xi=-p\operatorname{sgn}\dot{\xi}+\cos(\tau+\varphi) \tag{16}$$

The dot over ξ denotes differentiation with respect to τ.

Equation (16) is broken up into the following two equations:

$$\ddot{\xi}+\gamma_*^2\xi=-p+\cos(\tau+\varphi) \quad \text{at} \quad \dot{\xi}\geqslant 0 \tag{17}$$

$$\ddot{\xi}+\gamma_*^2\xi=p+\cos(\tau+\varphi) \quad \text{at} \quad \dot{\xi}\leqslant 0 \tag{18}$$

We shall seek the periodic solution with the period 2π for the case when there are no pauses of finite duration. Let the initial conditions at $\tau=0$ be

$$\xi=-\xi_0, \ \dot{\xi}=0 \tag{19}$$

at the beginning of the first stage when Eq. (17) becomes valid.

The general solution of Eq. (17) may be written in the following form:

$$\xi=a_1\cos(\gamma_*\tau+\psi_1)-\frac{p}{\gamma_*^2}+\frac{1}{\gamma_*^2-1}\cos(\tau+\varphi) \tag{20}$$

Hence

$$\dot{\xi}=-a_1\gamma_*\sin(\gamma_*\tau+\psi_1)-\frac{1}{\gamma_*^2-1}\sin(\tau+\varphi) \tag{21}$$

Inserting the initial conditions (19) into the relations obtained, we can write

$$a_1\cos\psi_1-\frac{p}{\gamma_*^2}+\frac{\cos\varphi}{\gamma_*^2-1}=-\xi_0 \tag{22}$$

$$-a_1\gamma_*\sin\psi_1-\frac{\sin\varphi}{\gamma_*^2-1}=0 \tag{23}$$

As the configuration is symmetric, we can write the terminal conditions for the first stage of the motion (after a half-period) which will also be the initial conditions for the second stage at $\tau=\pi$:

$$\xi=\xi_0, \ \dot{\xi}=0 \tag{24}$$

It should be noted that more complicated motions with the period 2π may take place when a finite interval of rest in the end position follows each stage of motion.

Inserting the initial conditions (24) into the relations (20) and (21) yields

$$a_1\cos(\pi\gamma_*+\psi_1)-\frac{p}{\gamma_*^2}-\frac{\cos\varphi}{\gamma_*^2-1}=\xi_0 \tag{25}$$

$$-a_1\gamma_*\sin(\pi\gamma_*+\psi_1)+\frac{\sin\varphi}{\gamma_*^2-1}=0 \tag{26}$$

Thus four equations (22), (23), (25) and (26) have been obtained which will serve to determine the integration constants a_1 and ψ_1, the initial displacement ξ_0 and the initial phase φ of the exciting force. We now proceed to the determination of these quantities. Adding equations (23) and (26) together, we obtain

$$\sin(\pi\gamma_* + \psi_1) + \sin\psi_1 = 0$$

Hence

$$2\sin\left(\frac{\pi\gamma_*}{2} + \psi_1\right)\sin\frac{\pi\gamma_*}{2} = 0$$

Since, generally speaking,

$$\sin\frac{\pi\gamma_*}{2} \neq 0$$

we have

$$\sin\left(\frac{\pi\gamma_*}{2} + \psi_1\right) = 0$$

Hence

$$\psi_1 = -\frac{\pi\gamma_*}{2} \tag{27}$$

Adding Eqs. (22) and (25) together and using expression (27), we get

$$a_1 = \frac{p}{\gamma_*^2}\sec\frac{\pi\gamma_*}{2} \tag{28}$$

Substituting the values of ψ_1 and a_1 from expressions (27) and (28) into Eq. (23), we obtain

$$\varphi = \sin^{-1}\left[\frac{p(\gamma_*^2 - 1)}{\gamma_*}\tan\frac{\pi\gamma_*}{2}\right] \tag{29}$$

Finally, from Eq. (22) and taking into account that $\xi_0 > 0$, we find

$$\xi_0 = \frac{1}{|\gamma_*^2 - 1|}\sqrt{1 - \frac{p^2(\gamma_*^2 - 1)^2}{\gamma_*^2}\tan^2\frac{\pi\gamma_*}{2}} \tag{30}$$

Expression (29) allows us to establish the following condition for the existence of the solution sought:

$$p \leqslant \frac{\gamma_*}{\left|(\gamma_*^2 - 1)\tan\frac{\pi\gamma_*}{2}\right|} \tag{31}$$

The general solution of differential equation (18) which describes the motion in the second stage can be presented in the form

$$\xi = a_2\cos(\gamma_*\tau + \psi_2) + \frac{p}{\gamma_*^2} + \frac{1}{\gamma_*^2 - 1}\cos(\tau + \varphi) \tag{32}$$

Hence

$$\dot{\xi} = -a_2\gamma_* \sin(\gamma_*\tau + \psi_2) - \frac{1}{\gamma_*^2 - 1}(\tau + \psi) \tag{33}$$

Inserting the initial conditions (24) into the last two expressions, we obtain the following equations from which the integration constants a_2 and ψ_2 can be determined:

$$a_2 \cos(\pi\gamma_* + \psi_2) + \frac{p}{\gamma_*^2} - \frac{\cos\varphi}{\gamma_*^2 - 1} = \xi_0 \tag{34}$$

$$-a_2\gamma_* \sin(\pi\gamma_* + \psi_2) + \frac{\sin\varphi}{\gamma_*^2 - 1} = 0 \tag{35}$$

From Eq. (35) and using the relation (29), we obtain

$$a_2 \sin(\pi\gamma_* + \psi_2) = \frac{p}{\gamma_*^2} \tan\frac{\pi\gamma_*}{2} \tag{36}$$

From Eq. (34) we find

$$a_2 \cos(\pi\gamma_* + \psi_2) = -\frac{p}{\gamma_*^2} \tag{37}$$

Dividing expression (36) by (37), we can write:

$$\tan(\pi\gamma_* + \psi_2) = -\tan\frac{\pi\gamma_*}{2}$$

whence

$$\pi\gamma_* + \psi_2 = -\frac{\pi\gamma_*}{2}$$

or

$$\psi_2 = -\frac{3\pi\gamma_*}{2} \tag{38}$$

and further

$$a_2 = -\frac{p}{\gamma_*^2} \sec\frac{\pi\gamma_*}{2} \tag{39}$$

The required solutions for both stages can now be written as follows:

$$\xi = -\frac{p}{\gamma_*^2} + \frac{p}{\gamma_*^2} \sec\frac{\pi\gamma_*}{2} \cos\left(\gamma_*\tau - \frac{\pi\gamma_*}{2}\right) +$$
$$+ \frac{1}{\gamma_*^2 - 1} \cos\left[\tau + \sin^{-1}\left(\frac{p(\gamma_*^2 - 1)}{\gamma_*} \tan\frac{\pi\gamma_*}{2}\right)\right] \quad \text{at} \quad 0 \leqslant \tau \leqslant \pi \tag{40}$$

$$\xi = \frac{p}{\gamma_*^2} - \frac{p}{\gamma_*^2} \sec\frac{\pi\gamma_*}{2} \cos\left(\gamma_*\tau - \frac{3\pi\gamma_*}{2}\right) +$$
$$+ \frac{1}{\gamma_*^2 - 1} \cos\left[\tau + \sin^{-1}\left(\frac{p(\gamma_*^2 - 1)}{\gamma_*} \tan\frac{\pi\gamma_*}{2}\right)\right] \quad \text{at} \quad \pi \leqslant \tau \leqslant 2\pi \tag{41}$$

The condition (31) is necessary but not sufficient. The sufficient condition is $\ddot{\xi} \geqslant 0$ at $\tau = 0$ whence, on the basis of the relations obtained, we get:

$$p \leqslant \frac{\gamma_*}{|\gamma_*^2 - 1| \sqrt{\gamma_*^2 + \tan^2 \frac{\pi\gamma_*}{2}}} \tag{42}$$

The above discussion clearly shows the possibilities of the method of fitting and the procedure to be followed in applying it. It should be borne in mind that only in a few cases is the situation so simple as in our example. The method of fitting leads, as a rule, to transcendental equations which can be solved only by numerical techniques and usually no general analytical relation for the motion stages can be obtained in explicit form.

Other examples of the application of the method of fitting will be given in Section 41.

21. Some Specific Features of Nonlinear Systems

An important negative feature of nonlinear systems is that the principle of superposition is not applicable to them. This is seen, in particular, from the fact that the sum of two or more solutions of such a system is not its general solution. It can also be seen from that the response of a nonlinear system to two or more simultaneously acting factors is not equal to the sum of the responses of the system to each of the factors applied separately.

Autonomous conservative nonlinear systems, generally speaking, are not isochronous (though there are special cases of isochronous nonlinear systems). This means that the frequency of the vibrations of such systems is dependent on the vibration swing. The natural frequency of systems with a hardening characteristic of the restoring force increases with increasing swing. If the restoring force has a softening characteristic, the natural frequency decreases with increase of vibration swing. Referring to Eq. (1), Sec. 19, this can be seen from formula (28), Sec. 19, which will be written now with only two terms being retained and expressions (3), Sec. 19, taken into account:

$$\omega = \omega_0 \left(1 + \frac{3c_1}{8c} a^2\right) \tag{1}$$

If $c_1 > 0$, i.e., the restoring force has a hardening characteristic, then $\partial\omega/\partial a > 0$. If $c_1 < 0$, then $\partial\omega/\partial a < 0$ (where a is the vibration half-swing).

In accordance with the above the resonant frequency of forced vibrations of nonlinear systems depends on the displacement swing. Figure 54*a* shows the "amplitude"[1] response curve of the system

described by the equation

$$\ddot{x} + \omega_0^2 x + \alpha^2 x^3 = u \cos \omega t \tag{2}$$

at $\alpha^2 > 0$, and the curve at $\alpha^2 < 0$ is given in Fig. 54*b*. In these figures the lines *6-7*, called skeleton curves, represent the relation (1) between the frequency of free vibrations and the "amplitude". The portions *4-3* correspond to unstable vibrations.

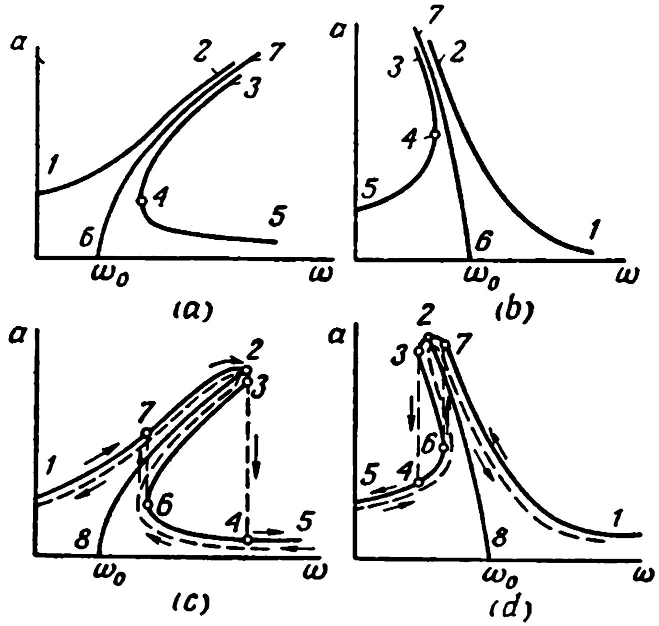


Figure 54

If linear dissipation of energy takes place in the system, we shall have, instead of Eq. (2),

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x + \alpha^2 x^3 = u \cos \omega t \tag{3}$$

The "amplitude" response curves furnished by Eq. (3) are shown in Fig. 54*c* for $\alpha^2 > 0$ and in Fig. 54*d* for $\alpha^2 < 0$.

We shall now use Fig. 54*c* to investigate the behaviour of the system at a sufficiently slow increase of the frequency ω, starting from zero. The increase of the half-swing value will follow the curve *1-2* and at point *2* it will reach a maximum (resonance). With further increase of the excitation frequency ω the half-swing value will drop to point *3* at which the tangent to the response curve is vertical. Any further increase in ω, no matter how small, will cause the half-swing to jump down from point *3* to point *4*. From this point on,

[1] The quotation marks are used to indicate that the quantity laid off along the ordinate is the half-swing rather than the amplitude.

with increasing ω, the half-swing value will diminish smoothly along the line *4-5*. If, starting from point *5*, the frequency ω is decreased at a sufficiently slow rate, the vibration half-swing will increase smoothly up to point *6*, at which the tangent to the response curve is vertical. Any further decrease, however small, in ω will cause the half-swing to jump up from point *6* to point *7*. From this point, with decreasing frequency, the vibration half-swing will smoothly diminish along the line *7-1*.

It should be noted that the response takes such a simple course when the excitation is provided by an ideal source of energy having an absolutely hard characteristic. This means that the excitation is completely independent of how the system reacts. Similar phenomena will occur in the case represented in Fig. 54*d* for oppositely directed changes in frequency.

As we see, there exist firstly critical frequencies corresponding to points *3*, *4* and *6*, *7* in Fig. 54*c* and *d* at which the vibration swing changes by a jump (sweep). Secondly, the amplitude response curves have portions *7-2* and *4-6* which can be attained when the excitation frequency changes only in one direction, i.e., only when the frequency increases or only when it decreases. Thirdly, there exist portions *3-6* which correspond to unstable states and therefore cannot be realized. Fourthly, when the excitation amplitude is changed (for instance, decreased), the amplitude response curves are shifted (see dotted lines in Fig. 54*c* and *d*) in such a way that the maximum moves along the line *2-8*, not leaving it. Hence the resonant frequency changes. It has already been pointed out (see Section 16) that there are some autonomous non-conservative nonlinear systems in which periodic vibrations with strictly definite swing and frequency (self-excited or self-induced vibrations) can set in. Such vibrations are maintained at the expense of the energy supplied from a non-oscillatory source.

Many nonlinear systems are capable of realizing qualitatively different motions, a very small change in one of the parameters being sufficient at the state boundaries to cause the system to jump from one type of motion to another differing qualitatively from the former. The vibration jump phenomenon discussed above is the simplest case of such a transition. Under the action of sinusoidally varying excitation the nonlinear system performs vibrations forming a broad spectrum. The fundamental vibration frequency may be lower than the excitation frequency by a factor of 2 or some other integer (subharmonic vibrations).

In many cases nonlinear systems display the phenomenon of frequency entrainment. This means that when two or more frequencies become sufficiently close to one another, the system vibrates at one frequency only. The self-synchronization of two or more coupled objects is a special case of the entrainment phenomenon.

CHAPTER 4

ENERGY RELATIONS

22. Transformation and Dissipation of Energy in Free Vibrations

The equation of motion of a conservative vibratory system can be written as follows:

$$m\ddot{x}+f(x)=0 \tag{1}$$

where $f(x)$ is the restoring force.

Using the identity

$$\ddot{x}\equiv\dot{x}\frac{d\dot{x}}{dx} \tag{2}$$

we can rewrite Eq. (1) in the form

$$m\dot{x}\,d\dot{x}+f(x)\,dx=0 \tag{3}$$

We now integrate this expression between the initial values of the variables contained in it

$$x=0,\quad \dot{x}=\dot{x}_m \tag{4}$$

and the running values:

$$\int\limits_{\dot{x}_m}^{\dot{x}} mu\,du+\int\limits_{0}^{x} f(z)\,dz=0 \tag{5}$$

Introducing the notation $\int f(x)\,dx=\Pi(x)$ and assuming $\Pi(0)=0$, we obtain

$$\frac{m\dot{x}^2}{2}+\Pi(x)=\frac{m\dot{x}_m^2}{2} \tag{6}$$

The integral (6) is the mathematical expression of the law of conservation of energy; the first term on the left-hand side is the running value of the kinetic energy T of the system, the second, the running value of the potential energy of the system. The constant on the right-hand side is equal to the total energy E of the system.

In fact, E is the kinetic energy of the system at the initial moment when its potential energy is zero.

As known, the level from which the potential energy is measured may be chosen arbitrarily. By putting $\Pi(0) = 0$ we have set the potential energy of the system to be zero at the position of stable equilibrium when $x = 0$. Maclaurin's power series representing $\Pi(x)$ thus takes the form (see Sec. 10)

$$\Pi(x) = \frac{\ddot{\Pi}(0)}{2!} x^2 + \frac{\dddot{\Pi}(0)}{3!} x^3 + \dots \tag{7}$$

Expression (6) may therefore be written in the following form:

$$T + \Pi = E \tag{8}$$

The potential energy of a vibrating system will attain its maximum value twice within a period

$$\Pi_{max} = E \tag{9}$$

at $x = x_{max}$ and $x = x_{min}$, i.e., when the displacement of the system from the equilibrium position to either side is the greatest.

At these moments the kinetic energy will have its minimum value

$$T_{min} = 0 \tag{10}$$

The system will pass through the equilibrium position twice within a period and at these moments its kinetic energy will reach a maximum

$$T_{max} = E \tag{11}$$

and its potential energy a minimum:

$$\Pi_{min} = 0 \tag{12}$$

Thus, the oscillations of the kinetic and potential energies occur with a swing E at a frequency double that of the vibration of the system under consideration.

This statement is best illustrated by the behaviour of a conservative linear single-degree-of-freedom system. It is described by Eq. (3), Sec. 6,

$$m\ddot{x} + cx = 0$$

In this case the potential energy is defined by the expression

$$\Pi = \frac{cx^2}{2} \tag{13}$$

If at $t = 0$ the conditions are similar to (4), i.e., we have $x = 0$, $\dot{x} = \dot{x}_a$, the general solution of Eq. (3), Sec. 6, will take, according to formula (10), Sec. 6, the following form:

$$x = \frac{\dot{x}_a}{\omega_0} \sin \omega_0 t \tag{14}$$

where ω_0 is given by expression (5), Sec. 6:

$$\omega_0 = \sqrt{\frac{c}{m}}$$

Since $\dot{x}_a = \omega_0 x_a$, we get, instead of expression (14),

$$x = x_a \sin \omega_0 t$$

whence

$$\dot{x} = \omega_0 x_a \cos \omega_0 t$$

The total energy of the system

$$E = \frac{m\dot{x}_a^2}{2} = \frac{m\omega_0^2 x_a^2}{2} = \frac{c x_a^2}{2} \tag{15}$$

Its kinetic energy is

$$T = E \cos^2 \omega_0 t = \frac{1}{2} E (1 + \cos 2\omega_0 t) \tag{16}$$

and the potential energy

$$\Pi = E \sin^2 \omega_0 t = \frac{1}{2} E (1 - \cos 2\omega_0 t) \tag{17}$$

The last two expressions show that the kinetic and potential energies oscillate in opposite phase at a frequency twice the natural frequency of the system. It follows that during one period of free vibrations of the system there occur two complete cycles of transformation of the kinetic into the potential energy and of the potential into the kinetic energy. Figure 55 illustrates the process.

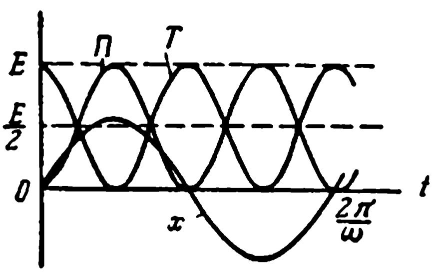


Figure 55

In the free vibrations of dissipative systems, besides the transformation of the kinetic into the potential energy and of the potential into the kinetic energy, there is dissipation of energy. Consider, for example, the system described by differential equation (27), Sec. 6:

$$m\ddot{x} + b\dot{x} + cx = 0$$

If we assume the initial conditions to be $x = 0$, $\dot{x} = \dot{x}_0$ at $t = 0$, then, in accordance with relations (30), (32), (33), and (34), Sec. 6, we shall have

$$x = \frac{\dot{x}_0}{\omega_1} e^{-ht} \sin \omega_1 t \tag{18}$$

where, as defined by formulas (29) and (31), Sec. 6,

$$h = \frac{b}{2m}; \quad \omega_1 = \sqrt{\omega_0^2 - h^2}$$

Differentiating expression (18) with respect to time, we obtain

$$\dot{x} = \dot{x}_0 \frac{\omega_0}{\omega_1} e^{-ht} \cos(\omega_1 t + \delta) \tag{19}$$

where δ is the loss angle; according to formulas (36), Sec. 6,

$$\cos \delta = \frac{\omega_1}{\omega_0}$$

With the conditions assumed $x = 0$ at the initial moment and the system possesses no potential energy; consequently the whole of its energy is in the form of kinetic energy. A similar situation will occur after any whole number of half-cycles (one cycle equals $2\pi/\omega_1$). After n cycles the energy of the system will be determined by the expression

$$E_n = \frac{m\dot{x}_n^2}{2} = \frac{m\dot{x}_0^2}{2} e^{-2\vartheta n} \tag{20}$$

where ϑ is the logarithmic decrement of vibrations which is defined by formula (42), Sec. 6, i.e.,

$$\vartheta = \frac{2\pi h}{\omega_1}$$

After $n+1$ cycles the energy will be

$$E_{n+1} = \frac{m\dot{x}_{n+1}^2}{2} = \frac{m\dot{x}_0^2}{2} e^{-2\vartheta(n+1)} \tag{21}$$

Hence the amount of energy dissipated during the $(n+1)$st cycle is

$$\Delta E_{n+1} = E_{n+1} - E_n = \frac{m\dot{x}_0^2}{2} e^{-2\vartheta n} (1 - e^{-2\vartheta}) \tag{22}$$

The total energy dissipated during the preceding n cycles

$$\sum_{i=1}^{n} \Delta E_i = \frac{m\dot{x}_0^2}{2} (1 - e^{-2\vartheta n}) \tag{23}$$

In a conservative linear multi-degree-of-freedom system specified by its normal coordinates (see Sec. 11), for each of its degrees of freedom defined by its normal coordinate and normal velocity there is observed the same picture of alternate transformations of the kinetic into the potential energy and vice versa as in a conservative linear single-degree-of-freedom system.

An essentially new phenomenon occurs in coupled systems. In these systems, apart from the energy transformations discussed above, there takes place an alternate exchange of energy (energy circula-

tion) between the degrees of freedom. Such a case was discussed in Section 11 for the free vibrations of a conservative system having two degrees of freedom shown schematically in Fig. 24*b*. The phenomenon was illustrated by Fig. 24*c* and *d*.

We now turn to an autonomous dissipative linear multi-degree-of-freedom system. Its equations of motion, according to formulas (10), (16), (20), and (22), Sec. 10, take the form

$$\sum_{j=1}^{n} a_{ij}\ddot{q}_j + \sum_{j=1}^{n} k_{ij}q_j + \sum_{j=1}^{n} b_{ij}\dot{q}_j = 0, \quad (i = 1, 2, 3, \ldots, n) \tag{24}$$

Let us find the rate of change of the total energy of the system, i.e., the derivative dE/dt. As defined by formulas (10) and (16), Sec. 10, the total energy of the system is

$$E = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\dot{q}_i\dot{q}_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}q_iq_j \tag{25}$$

Differentiating this expression with respect to time and making use of Eq. (24), we obtain

$$\frac{dE}{dt} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\dot{q}_i\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}\dot{q}_iq_j = \sum_{i=1}^{n} \dot{q}_i \left\{\sum_{j=1}^{n} a_{ij}\ddot{q}_j + \sum_{j=1}^{n} k_{ij}q_j\right\} =$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\dot{q}_i\dot{q}_j = -2\Phi \tag{26}$$

Thus the total energy of the system decreases (as evidenced by the minus sign) at a rate equal to twice the magnitude of the dissipative function.

23. The Power Necessary to Sustain Forced Vibrations. Energy Circulation Between the Source and the Vibrating System

Consider a single-degree-of-freedom system described by Eq. (16), Sec. 7,

$$m\ddot{x} + b\dot{x} + cx = F_a \cos \omega t$$

The periodic solution of the equation is given by expressions (19) through (21), Sec. 7:

$$x = x_a \cos(\omega t - \varphi)$$

where

$$x_a = \frac{F_a}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}}$$

$$\varphi = \tan^{-1}\frac{2h\omega}{\omega_0^2 - \omega^2}$$

The velocity

$$\dot{x} = -\dot{x}_a \sin(\omega t - \varphi) = -\omega x_a \sin(\omega t - \varphi)$$

The energy needed to sustain the vibrations during one period is defined by the expression

$$A = \int_0^{\frac{2\pi}{\omega}} F\dot{x}\,dt \tag{1}$$

where, according to (1), Sec. 7,

$$F = F_a \cos \omega t$$

Insertion of the values of the exciting force and velocity into the integrand gives

$$A = -F_a x_a \omega \int_0^{\frac{2\pi}{\omega}} \cos \omega t \sin(\omega t - \varphi)\,dt$$

Hence

$$A = \pi F_a x_a \sin \varphi \tag{2}$$

The expression of the mean power developed by the energy source is now readily written:

$$N_{mean} = \frac{\omega A}{2\pi} = \frac{1}{2} F_a x_a \omega \sin \varphi \tag{3}$$

Substituting the value of x_a and

$$\sin \varphi = \frac{2h\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}}$$

we obtain

$$N_{mean} = \frac{F_a^2 \omega^2 h}{m\left[(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2\right]} \tag{4}$$

Making use of the obvious relation

$$x_a = \frac{F_a \cos \varphi}{m(\omega_0^2 - \omega^2)}$$

we can reduce formula (3) for the mean power to the form

$$N_{mean} = \frac{F_a^2 \omega \sin 2\varphi}{4m(\omega_0^2 - \omega^2)} \tag{5}$$

If the exciting force in expression (1) is replaced by its equivalent on the left-hand side of differential equation (16), Sec. 7, then, taking into account expression (3), we obtain

$$N_{mean} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} m\ddot{x}\dot{x}\,dt + \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} b\dot{x}^2\,dt + \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} cx\dot{x}\,dt \tag{6}$$

Because of the mutual orthogonality of x, $\dot{x}$ and $\dot{x}$, $\ddot{x}$ within one period of vibration the first and third integrals vanish. This means that with periodic vibrations the exciting mechanism need not develop any power to overcome the inertia and elastic forces or other potential forces (it is the mean power that is meant here, not its instantaneous values). Consequently, power is required only to overcome the dissipative force. Integrating the second term on the right-hand side of expression (6), we obtain, after necessary transformations, relation (5) already known to us.

The instantaneous or running value of the power developed by the source is equal to the product of the instantaneous values of the exciting force and vibration velocity:

$$N = F\dot{x} \tag{7}$$

whence

$$N = -F_a x_a \omega \cos \omega t \sin (\omega t - \varphi)$$

or

$$N = \frac{1}{2} F_a x_a \omega \sin \varphi - \frac{1}{2} F_a x_a \omega \sin (2\omega t - \varphi) \tag{8}$$

Using expression (3), we can also write:

$$N = N_{mean} - \frac{1}{2} F_a x_a \omega \sin (2\omega t - \varphi) \tag{9}$$

Thus, the instantaneous power N developed by the energy source is a sinusoidally varying function of time which oscillates at a frequency twice the frequency ω of the vibrations of the system about the mean value N_{mean}. Since $\sin \varphi < 1$, the amplitude of power oscillation $0.5\ F_a x_a \omega$ exceeds the constant component (mean value) $0.5\ F_a x_a \omega \sin \varphi$. Owing to this the instantaneous power is a function which changes its sign four times during one period $2\pi/\omega$ of vibration. So twice during one period of vibration the energy flows from the source to the system (when the power developed by the system is positive), and twice it flows back from the vibrating system to the energy source (when the power developed by the source is negative).

The maximum (positive) value of the power

$$N_{max} = \frac{1}{2} F_a x_a \omega (1 + \sin \varphi) \tag{10}$$

The minimum (negative) value of the power

$$N_{min} = -\frac{1}{2} F_a x_a \omega (1 - \sin \varphi) \tag{11}$$

Obviously at $\pi > \varphi > 0$ (when $h > 0$)

$$|N_{max}| > |N_{min}|$$

In the case of $\varphi = \pi/2$ (which means that $\omega = \omega_0$ or the unattainable limiting value $h = \infty$)

$$N_{min} = 0, \quad N_{max} = F_a x_a \omega$$

i.e., the instantaneous power in this exceptional case always exceeds zero or is zero. Figure 56*a* shows the curve of instantaneous power for the general case.

Making use of the identity transformation

$$\sin(2\omega t - \varphi) = \sin(2\omega t - 2\varphi + \varphi) =$$
$$= \sin\varphi\cos(2\omega t - 2\varphi) + \cos\varphi\sin(2\omega t - 2\varphi)$$

we obtain from relation (8) the expression of the instantaneous power in the form of a sum of two terms different from those in (8):

$$N = \frac{1}{2}F_a x_a \omega \sin\varphi\,[1 - \cos(2\omega t - 2\varphi)] - \frac{1}{2}F_a x_a \omega \times \cos\varphi \sin(2\omega t - 2\varphi) \qquad (12)$$

where the first term on the right-hand side remains non-negative all the time and oscillates about N_{mean} with the amplitude N_{mean}. The second term oscillates about zero. Figure 56*b* shows the two terms.

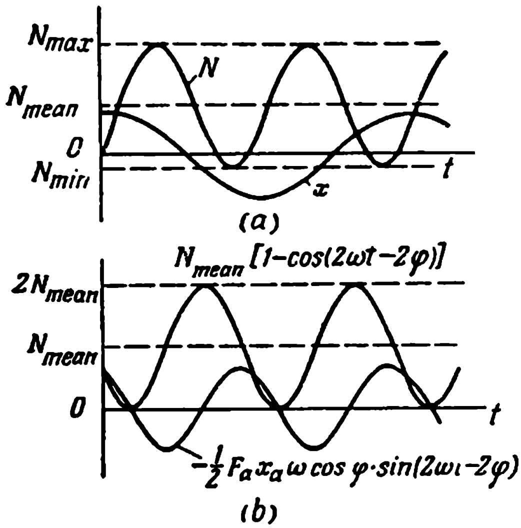


Figure 56

Consider now expression (7) in the form

$$N = (m\ddot{x} + b\dot{x} + cx)\,\dot{x} \qquad (13)$$

It is easily verified that the first term on the right-hand side of expression (12) corresponds to the power developed on the dissipative element of the system,

$$\frac{1}{2}F_a x_a \omega \sin\varphi\,[1 - \cos(2\omega t - 2\varphi)] = b\dot{x}^2 \qquad (14)$$

and the second term corresponds to the sum of the powers developed on the mass and elastic elements (reactive elements) of the system,

$$-\frac{1}{2}F_a x_a \omega \cos\varphi \sin(2\omega t - 2\varphi) = (m\ddot{x} + cx)\,\dot{x} \qquad (15)$$

it vanishes at $\varphi = \pi/2$, i.e., with $\omega = \omega_0$.

The amplitude value of the varying part of the first term on the right-hand side of expression (12) is called the *active power*

$$N_{act} = N_{mean} = \frac{1}{2} F_a x_a \omega \sin \varphi = \frac{1}{2} F_a \dot{x}_a \sin \varphi \tag{16}$$

and the amplitude value of the second term is known as the *reactive power*

$$N_{react} = \frac{1}{2} F_a x_a \omega \cos \varphi = \frac{1}{2} F_a \dot{x}_a \cos \varphi \tag{17}$$

One can make use of the effective (root-mean-square) values of the exciting force and vibration velocity

$$\left.\begin{aligned} F_{eff} &= \sqrt{\frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} F^2 \, dt} = \frac{\sqrt{2}}{2} F_a \\ \dot{x}_{eff} &= \sqrt{\frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \dot{x}^2 \, dt} = \frac{\sqrt{2}}{2} \dot{x}_a \end{aligned}\right\} \tag{18}$$

and then express the active and reactive power as follows:

$$N_{act} = F_{eff} \dot{x}_{eff} \sin \varphi, \quad N_{react} = F_{eff} \dot{x}_{eff} \cos \varphi \tag{19}$$

24. The Maximum Mean Power of a Vibration Generator

One of the difficult problems in designing new types of vibration machines is the calculation of the power required to sustain the vibrations. The computations should be based on the given dissipative resistances but one has usually only a rather vague idea of them. Generally no standard methods of calculating the dissipative parameters of a particular design are available. Even experimental investigations and testing of machines do not always provide an answer. This is why gross errors in the evaluation of the driving power are far from exceptional in designing new types of vibration machines.

In many cases the situation is complicated by the fact that the factors determining the energy dissipation in vibrations are not stable and vary within rather wide limits. In some cases these variations are a consequence of the operation of a machine, in others they result from random processes.

Not infrequently the only reliable criterion proves to be the maximum of the mean power of the vibration generator. Moreover, in many cases the maximum power is realized in practice and the criterion becomes necessary and sufficient.

The problem of finding the maximum mean power of the vibration generator can be formulated as follows: one must first establish at which value of the resistance coefficient (or some other parameter unambiguously expressible in terms of this coefficient) the vibration generator will develop its maximum power on the condition that other parameters of the system (mass, stiffness, and the amplitude and frequency of the exciting force) remain unchanged and then find the maximum value of the mean power.

For a single-degree-of-freedom linear system the problem is readily solved by considering the relation (5), Sec. 23. The right-hand side of this relation reaches its maximum value either if $\sin 2\varphi = 1$ at $\omega_0 > \omega$ or if $\sin 2\varphi = -1$ at $\omega_0 < \omega$. The maximum value is attained at $\varphi_m = \pi/4$ in the first case, and at $\varphi_m = 3\pi/4$ in the second. In both cases the maximum mean power of the vibration generator is given by

$$\max N_{mean} = \frac{F_a^2 \omega}{4m\,|\omega_0^2 - \omega^2|} \tag{1}$$

The damping coefficient at which the power reaches its maximum can be determined from the relation

$$|\tan \varphi_m| = \frac{2h_m\omega}{|\omega_0^2 - \omega^2|} = 1$$

whence

$$h_m = \frac{|\omega_0^2 - \omega^2|}{2\omega} \tag{2}$$

As the coefficient of resistance $b = 2hm$ we obtain

$$b_m = \frac{m\,|\omega_0^2 - \omega^2|}{\omega} = \frac{|c - m\omega^2|}{\omega} \tag{3}$$

The corresponding damping ratio defined by formula (43), Sec. 6, is

$$\beta_m = \frac{|1 - \gamma^2|}{2\gamma} \tag{4}$$

where $\gamma = \omega/\omega_0$ according to the second of the expressions (3), Sec. 13.

The quality factor, by formula (44), Sec. 6,

$$Q_m = \frac{\gamma}{|1 - \gamma^2|} \tag{5}$$

The logarithmic decrement of vibrations (it has meaning at $\beta < 1$) is, according to Table 1,

$$\vartheta = \frac{2\pi}{\sqrt{\dfrac{4\gamma^2}{(1-\gamma^2)^2} - 1}} \tag{6}$$

The loss angle (it has meaning at $\beta \leqslant 1$) from the same table

$$\delta_m = \sin^{-1} \frac{|1-\gamma^2|}{2\gamma} \tag{7}$$

Expression (1) is not valid at $\omega_0 = \omega$ as in this case $\varphi = \pi/2$. The right-hand side of expression (5), Sec. 23, becomes indeterminate at $\varphi = \pi/2$ and $\omega_0 = \omega$; its value is found by substituting

$$\sin 2\varphi = \frac{4h\omega(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}$$

Thus, with $\omega_0 = \omega$

$$N_{mean} = \frac{F_a^2}{4mh} = \frac{F_a^2}{2b} \tag{8}$$

and the maximum, $\max N_{mean} = \infty$, is reached at $h_m = 0$ (b_m 0).

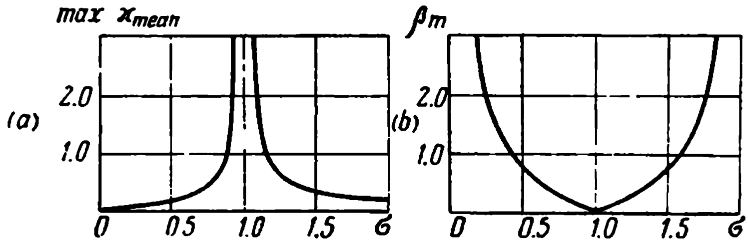


Figure 57

Introducing a dimensionless parameter which is proportional to the power (we shall call it the dimensionless power)

$$\varkappa = \frac{m\omega}{F_a^2} N \tag{9}$$

we can rewrite expression (5), Sec. 23, in the form

$$\varkappa_{mean} = \frac{\gamma^2 \sin 2\varphi}{4(1-\gamma^2)} = \frac{\sin 2\varphi}{4(\gamma_*^2 - 1)} \tag{10}$$

where $\gamma_* = 1/\gamma$ according to the second of the expressions (6), Sec. 13.

Expression (1) may then assume the form

$$\max \varkappa_{mean} = \frac{\gamma^2}{4|1-\gamma^2|} = \frac{1}{4|\gamma_*^2 - 1|} \tag{11}$$

Finally, relation (8) which is valid at $\gamma = \gamma_* = 1$ takes the form

$$\varkappa_{mean} = \frac{1}{4\beta} \tag{12}$$

Figure 57*a* shows the relation between $\max \varkappa_{mean}$ and σ calculated from formula (10), the value of σ being determined from

equalities (15), Sec. 13:

$$\sigma=\begin{cases}\gamma & \text{at} \quad 0\leqslant\gamma\leqslant 1\\ 2-\dfrac{1}{\gamma} & \text{at} \quad 1\leqslant\gamma<\infty\end{cases}$$

Figure 57*b* shows β_m against σ plotted from formula (4). The parameters calculated above are tabulated in Table 8.

The use of the criterion of maximum power is complicated by the fact that dissipative resistances are often substantially nonlinear

TABLE 8

Parameter	Value of parameter at		
	$\omega<\omega_0$	$\omega>\omega_0$	$\omega=\omega_0$
max N_{mean}	$\dfrac{F_a^2\omega}{4m\,\lvert\omega_0^2-\omega^2\rvert}$		∞
max $\varkappa_{mean}$	$\dfrac{\gamma^2}{4\lvert 1-\gamma^2\rvert}$		∞
φ_m	$\dfrac{\pi}{4}$	$\dfrac{3\pi}{4}$	$\dfrac{\pi}{2}$
b_m	$\dfrac{\lvert c-m\omega^2\rvert}{\omega}$		0
h_m	$\dfrac{\lvert\omega_0^2-\omega^2\rvert}{2\omega}$		0
β_m	$\dfrac{\lvert 1-\gamma^2\rvert}{2\gamma}$		0
Q_m	$\dfrac{\gamma}{\lvert 1-\gamma^2\rvert}$		∞
ϑ_m	$\dfrac{2\pi}{\sqrt{\dfrac{4\gamma^2}{(1-\gamma^2)^2}-1}}$		0
δ_m	$\sin^{-1}\dfrac{\lvert 1-\gamma^2\rvert}{2\gamma}$		0

functions of velocity whose form is in general unknown. It is therefore important to establish whether the results obtained for linear systems may be used with systems whose dissipative resistances are nonlinear.

To this end we shall consider a set of laws of energy dissipation where the dissipative force $-B$ is an odd function of velocity.

The differential equation of the forced vibrations of a single-degree-of-freedom system will then take the following form:

$$m\frac{d^2x}{dt^2}+B\left(\frac{dx}{dt}\right)+cx=F_a\cos\omega t \tag{13}$$

We now introduce dimensionless variables (4), Sec. 13, dropping the asterisks which are of no use here:

$$\tau=\omega t,\quad \xi=\frac{m\omega^2}{F_a}x \tag{14}$$

We also introduce the dimensionless parameters

$$\beta_*=\frac{1}{2F_a}B\left(\frac{F_a}{m\omega}\right),\quad \gamma_*=\frac{\omega_0}{\omega} \tag{15}$$

The parameter γ_* is analogous to the parameter defined by the second of the expressions (6), Sec. 13, and the parameter β_* is the generalized form of the parameter defined by the first of the expressions (6), Sec. 13, and becomes identical with it in the special case of a linear dissipative resistance when $B(\dot{x})=b\dot{x}$. Introducing the notation

$$f(\dot{\xi})=\frac{B\left(\frac{F_a\dot{\xi}}{m\omega}\right)}{B\left(\frac{F_a}{m\omega}\right)} \tag{16}$$

and inserting the relations (14) through (16) into Eq. (13), we obtain

$$\ddot{\xi}+2\beta_*f(\dot{\xi})+\gamma_*^2\xi=\cos\tau \tag{17}$$

The solution of this equation corresponding to the periodic vibrations of a system at the frequency of the exciting factor can be represented by the series (cf. Section 4)

$$\xi=\sum_{n=1}^{\infty}c_n\cos(n\tau-\varphi_n) \tag{18}$$

where c_n and φ_n are the amplitude and the initial phase of the nth harmonic, respectively.

Since the exciting factor has the frequency of the first harmonic and the sine and cosine functions of multiple frequencies within the period are mutually orthogonal, it will do work whose mean value is different from zero only on the first harmonic of the vibrations. Consequently, in order to calculate the mean (active) power

it is sufficient to determine only the first harmonic of the series (18). In view of this we set

$$\xi = c_1 \cos(\tau - \varphi_1) \tag{19}$$

Hence

$$\dot{\xi} = -c_1 \sin(\tau - \varphi_1), \quad \ddot{\xi} = -c_1 \cos(\tau - \varphi_1) \tag{20}$$

As $f(\dot{\xi})$ is an odd function of $\dot{\xi}$ it can be expanded in the following series:

$$f(\dot{\xi}) = \sum_{n=1}^{\infty} p_n \sin n(\tau - \varphi_1) \tag{21}$$

The amplitude of the first harmonic of the series can be determined from the second of formulas (2), Sec. 4:

$$p_1 = -\frac{1}{\pi} \int_0^{2\pi} f[c_1 \sin(\tau - \varphi_1)] \sin(\tau - \varphi_1)\, d\tau = -\Phi(c_1) \tag{22}$$

We insert now the expressions (19) and (20) into Eq. (17) and replace $f(\dot{\xi})$ by the first harmonic in accordance with expressions (21) and (22):

$$-c_1 \cos(\tau - \varphi_1) - 2\beta_* \Phi(c_1) \sin(\tau - \varphi_1) + \gamma_*^2 c_1 \cos(\tau - \varphi_1) \equiv \cos\tau \tag{23}$$

Substituting successively the values of $\tau = \varphi_1$ and $\tau = \varphi_1 - \pi/2$ into the identity (23), we obtain

$$\sin\varphi_1 = 2\beta_* \Phi(c_1), \quad \cos\varphi_1 = c_1(\gamma_*^2 - 1) \tag{24}$$

whence

$$F(c_1, \beta_*) = c_1^2(\gamma_*^2 - 1)^2 + 4\beta_*^2 \Phi^2(c_1) - 1 = 0 \tag{25}$$

The values of c_1 and φ_1 furnished by these equations constitute the first approximation the error of which is the less the nearer the motion considered approaches the sinusoidal vibration.

The dimensionless power averaged over the period is defined by the expression

$$\varkappa_{mean} = \frac{1}{2\pi} \int_0^{2\pi} \dot{\xi} \cos\tau\, d\tau \tag{26}$$

Substituting into (26) the value of the first harmonic of $\dot{\xi}$ defined by the first of expressions (20), we obtain

$$\varkappa_{mean} = \frac{c_1}{2} \sin\varphi_1 \tag{27}$$

or, making use of the first of expressions (24),

$$\varkappa_{mean} = \beta_* c_1 \Phi(c_1) \tag{28}$$

We shall determine now max $\varkappa_{mean}$. This maximum value is reached at

$$\frac{d\varkappa_{mean}}{d\beta_*}=0 \tag{29}$$

The amplitude c_1 depends on β_*, and the relation $F(c_1, \varphi_*)=0$ is expressed by Eq. (25). Therefore

$$\frac{d\varkappa_{mean}}{d\beta_*}=\frac{\partial\varkappa_{mean}}{\partial\beta_*}-\frac{\partial\varkappa_{mean}}{\partial c_1}\cdot\frac{\dfrac{\partial F}{\partial\beta_*}}{\dfrac{\partial F}{\partial c_1}} \tag{30}$$

Inserting into (30) the values of $\partial\varkappa_{mean}/\partial\beta_*$ and $\partial\varkappa_{mean}/\partial c_1$ calculated from Eq. (28) and those of $\partial F/\partial\beta_*$ and $\partial F/\partial c_1$ found from Eq. (25) and equating the result to zero in accordance with Eq. (29), we obtain

$$\beta_{*m}=\frac{c_{1m}\,|\gamma_*^2-1|}{2\Phi(c_{1m})} \tag{31}$$

Substituting expression (31) into Eq. (25), we find:

$$c_{1m}=\frac{1}{\sqrt{2}\,|\gamma_*^2-1|} \tag{32}$$

Making use of the results (31) and (32), we obtain, from expression (28), the required quantity:

$$\max\varkappa_{mean}=\frac{1}{4\,|\gamma_*^2-1|} \tag{33}$$

Thus, we have obtained the same relation as (11) which was derived for a linear system. This means that for systems with nonlinear dissipation the maximum mean (active) power calculated to a first approximation which can be realized by the vibration generator is independent of the form of the law of energy dissipation and is defined by the same expression as in the case of a linear system.

It follows from expressions (24) that the necessary conditions that must be observed are:

$$\Phi(c_1)\leqslant\frac{1}{2\beta_*},\quad c_1\leqslant\frac{1}{|\gamma_*^2-1|} \tag{34}$$

These conditions are satisfied at $c_1=c_{1m}$ and $\beta_*=\beta_{*m}$. That this is true of the second condition (34) follows from relation (32). Inserting the value of c_{1m} from relation (32) into expression (31), we obtain

$$\Phi(c_{1m})=\frac{1}{2\sqrt{2}\beta_{*m}}$$

i.e., the first of the conditions (34) is also satisfied.

If we consider the set of the laws of energy dissipation in vibrations for which the dissipative force can be expressed as a single term representing an odd power-function of velocity,

$$B = -b\left|\frac{dx}{dt}\right|^{\alpha}\operatorname{sgn}\frac{dx}{dt}, \quad (0 \leqslant \alpha < \infty) \tag{35}$$

then from what has been said we obtain

$$\left.\begin{aligned} -B &= \frac{bF_a^{\alpha-1}}{2(m\omega)^{\alpha}}; \quad f(\dot{\xi}) = |\dot{\xi}|^{\alpha}\operatorname{sgn}\dot{\xi} \\ \Phi(c_1) &= \frac{c_1^{\alpha}\Gamma(\alpha+1)}{2^{\alpha-1}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+3}{2}\right)} \\ \beta_{*m} &= \frac{|\gamma_*^2-1|^{\alpha}2^{(3\alpha-5)/2}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+3}{2}\right)}{\Gamma(\alpha+1)} \end{aligned}\right\} \tag{36}$$

where $\Gamma(z)$ is the Eulerian gamma-function.

In the special case $\alpha = 0$, i.e., when the dissipative resistance is the so-called dry or Coulomb friction, we have

$$\left.\begin{aligned} \beta_* &= \frac{b}{2F_a}; \quad f(\dot{\xi}) = \operatorname{sgn}\dot{\xi} \\ \Phi(c_{1m}) &= \frac{4}{\pi}; \quad \beta_{*m} = \frac{\pi}{8\sqrt{2}} \end{aligned}\right\} \tag{37}$$

In Section 20 an exact periodic solution was obtained for this case with the period 2π (motion without pauses). The dimensionless mean power can be determined from the formula

$$\varkappa_{mean} = \frac{2p\xi_0}{\pi} \tag{38}$$

where ξ_0, the maximum displacement from the equilibrium position, is found from (31), Sec. 20, and $p = 2\beta_*$. Substituting the values of ξ_0 and p into the right-hand side of expression (38), we obtain

$$\varkappa_{mean} = \frac{4\beta_*}{\pi|\gamma_*^2-1|}\cdot\sqrt{1-\frac{4\beta_*^2(\gamma_*^2-1)^2}{\gamma_*^2}\tan^2\frac{\pi\gamma_*}{2}} \tag{39}$$

This expression reaches its maximum at

$$\beta_{*m} = \frac{\gamma_*}{2\sqrt{2}\left|(\gamma_*^2-1)\tan\frac{\pi\gamma_*}{2}\right|} \tag{40}$$

and this maximum is determined from the following expression:

$$\max\varkappa_{mean} = \frac{\gamma_*}{\pi(\gamma_*^2-1)^2\left|\tan\frac{\pi\gamma_*}{2}\right|} \tag{41}$$

Expression (40) satisfies condition (31), Sec. 20. In order to satisfy also condition (42), Sec. 20, the inequality

$$\gamma_* < \left| \tan \frac{\pi\gamma_*}{2} \right|$$

must be fulfilled.

The values of max $\varkappa_{mean}$ calculated from formulas (41) and (33) and the percentage errors in formula (33) are tabulated in Table 9.

TABLE 9

Values of max $\varkappa_{mean}$ at $\gamma_*=$								
0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
From formula (41)								
0.203	0.206	0.214	0.227	0.247	0.282	0.337	0.438	0.639
From formula (33)								
0.250	0.252	0.260	0.274	0.296	0.333	0.390	0.490	0.690
Errors in formula (33), per cent								
+23.0	+22.3	+21.4	+20.7	+19.8	+18.1	+15.8	+11.9	+8.0

The data in Table 9 show that the approximate formula (33) yields good results even in the case of such a nonlinear dissipation as dry friction.

CHAPTER 5

THE DYNAMICS OF VIBRATION GENERATORS

25. Generation of Single-Frequency Excitations

The alternating exciting forces developed by centrifugal vibration generators are centrifugal inertia forces produced by the rotation of unbalanced rotors called unbalances (also called *unbalanced masses* or *eccentric weights*). Practical demands necessitate the wide use of vibration generators whose rotors are statically unbalanced. However there are also dynamically unbalanced and combined types of vibration machines.

The absolute magnitude, spatial orientation and character of time variation of the exciting forces and moments of centrifugal (unbalanced-mass) vibration generators depend on the motion of the working member vibrated by means of the generator and on the properties of the driving motor which rotates the unbalances. The motion of the working member is determined, in particular, by the position of the generator, the type of connections between the working member and the generator, and by interactions with the medium with which the working member is in contact either directly or via intermediate elements. Hence it is necessary to study the dynamics of the whole system comprising the drive, vibration generator proper, working member and external medium as well as the connections and intermediate elements.

We shall here consider several schemes of the generation of excitation, assuming that the vibration generator body and the axes of rotation of the rotors are fixed and the unbalanced masses are not acted upon by any forces or moments, i. e., that they rotate at a constant angular velocity. Out of a multitude of possible schemes only the simplest ones covering the greater part of practical applications have been selected.

Consider first plane diagrams with statically unbalanced rotors (Fig. 58*a*-*l*). Here the centrifugal forces of all the unbalanced masses are coplanar and lie in the plane of the drawing. The vectors of the moments of forces, if there arise any, are perpendicular to the plane of the drawing. If there is only one unbalance (Fig. 58*a*), we have a circular exciting force (a rotating exciting force of constant

modulus). The hodograph of the exciting-force vector is a circle (Fig. 58*b*). The numbers designate the successive positions of the unbalance and the corresponding exciting-force vectors. If we denote by K the static mass moment of the unbalance with respect to the

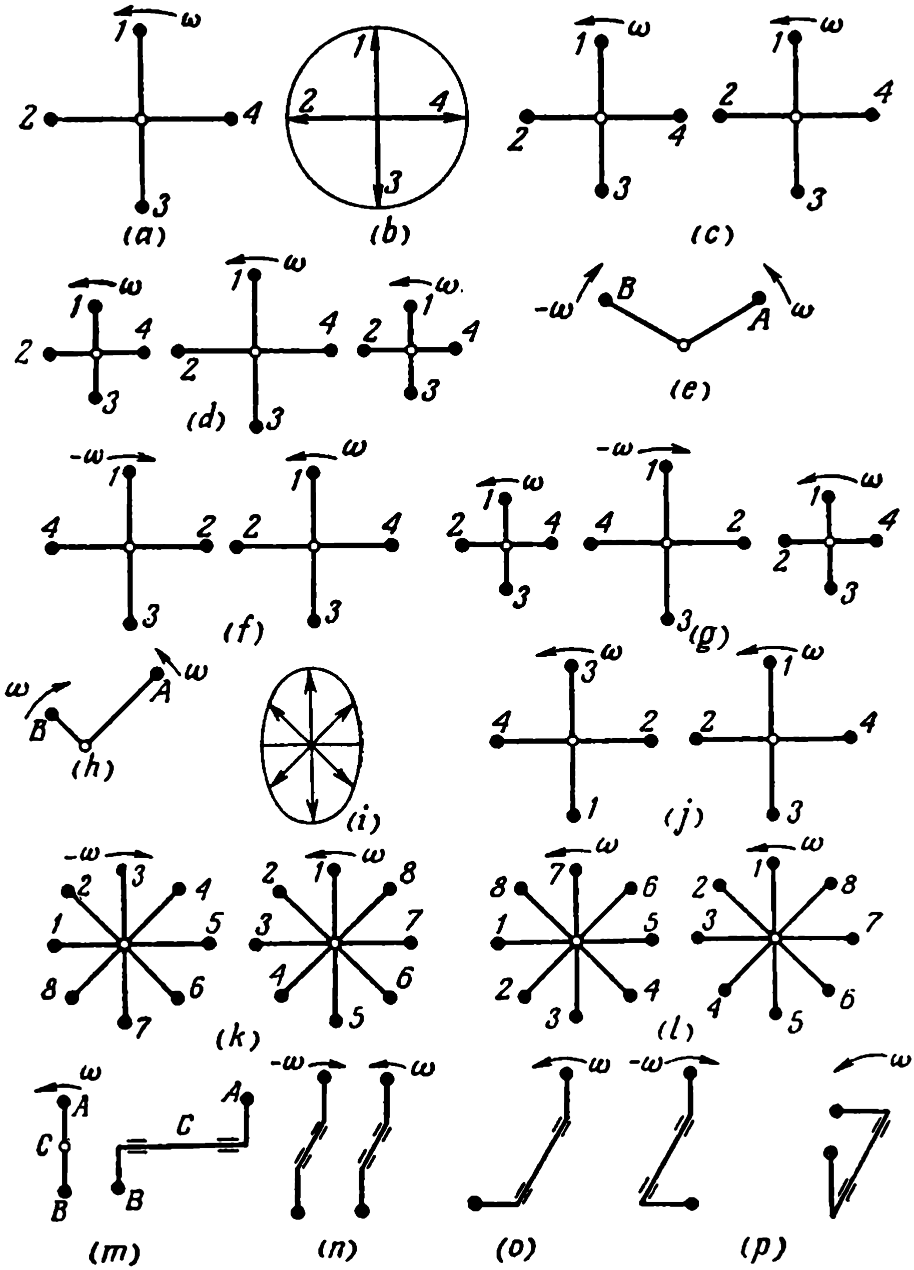


Figure 58

axis of rotation, then the exciting-force modulus is determined from the expression

$$F_a = K\omega^2 \tag{1}$$

where ω is the angular velocity of rotation of the unbalanced mass; the vector itself can be represented in the form

$$F = F_a e^{i\omega t} \tag{2}$$

in accordance with expression (12), Sec. 2, if the unbalanced mass rotates in the positive sense, and in the form

$$F = F_a e^{-i\omega t} \tag{3}$$

if it rotates in the negative sense.

A circular exciting force reduced to the midpoint of the generator, which usually coincides with its centre of gravity, can be also produced in more complicated systems, for instance, in the presence of two equal unbalances rotating in phase in the same sense as shown in Fig. 58*c*. If the static mass moment of each of the unbalances with respect to the axis of rotation is K, then the exciting-force modulus $F_a = 2K\omega^2$. The generator midpoint is at the middle of the segment connecting the axes of rotation of the unbalanced masses. In the case of three unbalances, Fig. 58*d*, the circular exciting force reduced to the midpoint situated on the axis of the central eccentric weight can be produced by rotating the three eccentric weights in phase in the same sense, placing the axes of the unbalances on the sides on the same straight line with the axis of the central mass and symmetrically to it and by making the static mass moments of the side unbalances equal.

A sinusoidally varying exciting force with unchanging direction (directed exciting force) can be generated by rotating two equal unbalances A and B about a common axis (Fig. 58*e*) in opposite senses at the same (as to modulus) angular velocity. The exciting-force line of action intersects the axis of rotation. If the static mass moment of each unbalance is K, then the amplitude of the exciting force $F_a = 2K\omega^2$. In accordance with the closing remark of Section 2 the exciting force is defined in this case by the expression

$$F = \frac{1}{2} F_a e^{i\omega t} + \frac{1}{2} F_a e^{-i\omega t} = F_a \cos \omega t \tag{4}$$

under the assumption that the initial phase is zero. A similar result can be obtained if each of the two unbalanced masses rotates about an individual axis (Fig. 58*f*) with the line of action of the force passing through the middle point of the straight-line segment between the axes of the unbalanced masses.

If the directed exciting force is to be produced by making use of three unbalanced masses, as shown schematically in Fig. 58*g*, with the axes of the side unbalanced masses placed symmetrically

relative to the axis of the central mass, then the side masses whose static mass moment is equal to half of the static mass moment K of the central mass must rotate in phase in the same sense. The central unbalanced mass must rotate in the opposite sense at the same angular velocity (as to modulus). The exciting-force line of action will pass through the axis of the central unbalanced mass. Its amplitude $F_a = 2K\omega^2$.

An elliptical exciting force can be realized by using two unbalances A and B of different static mass moments (Fig. 58*h*) rotating about the same axis in opposite senses at the same angular velocity (as to modulus). The hodograph of the exciting force is shown in Fig. 58*i*. The force is applied to the axis of rotation.

A sinusoidally varying exciting moment can be generated by using two equal unbalanced masses rotating in antiphase in the same directions (each about its individual axis) as shown in Fig. 58*j*. In this case the exciting moment is defined by the expression

$$M = M_a \cos \omega t \tag{5}$$

and its amplitude

$$M_a = Kl\omega^2 \tag{6}$$

where K = static mass moment of one unbalance
l = distance between the axes of rotation of the unbalances.

A directed exciting force together with an oscillating exciting moment can be realized by making use of the arrangement shown in Fig. 58*f* but with different initial phases as illustrated in Fig. 58*k*.

A circular exciting force together with an oscillating exciting moment can be obtained by using the arrangement illustrated in Fig. 58*c* but with different initial phases (cf. Fig. 58*l*).

Of the spatial arrangements consider in the first place the case of purely dynamic unbalance of one rotor; we represent it schematically by two equal unbalanced masses A and B fixed in antiphase on the two ends of shaft C. Two projections of the arrangement are pictured in Fig. 58*m*. A rotating exciting moment of constant modulus (circular moment) is generated. The vector of this moment rotates in the plane of the projection on the left.

Two such identical rotors with parallel shafts rotating in opposite senses at the same angular velocity (as to modulus) and phased as pictured in the axonometric diagram (Fig. 58*n*) generate a sinusoidal exciting moment whose vector lies in the plane of the axes of the rotors and is perpendicular to them.

If the rotor is unbalanced both statically and dynamically as shown in the axonometric diagram in Fig. 58*o*, then a circular moment and a circular force will be generated together.

With equal unbalanced masses the circular force is applied at the midpoint of the axis.

Two rotors (Fig. 58*o*) rotating about parallel axes in opposite senses at the same angular velocity (as to modulus) and phased as shown in Fig. 58*p* generate a sinusoidally varying dynamic helix, i.e., sinusoidal in-phase (or in-antiphase) exciting force and exciting moment whose vectors are parallel.

26. Reduction of Stiffnesses, Resistance Coefficients and Masses

Elastic elements in one-dimensional arrangements may be grouped in parallel or in series. Elastic elements grouped in parallel are

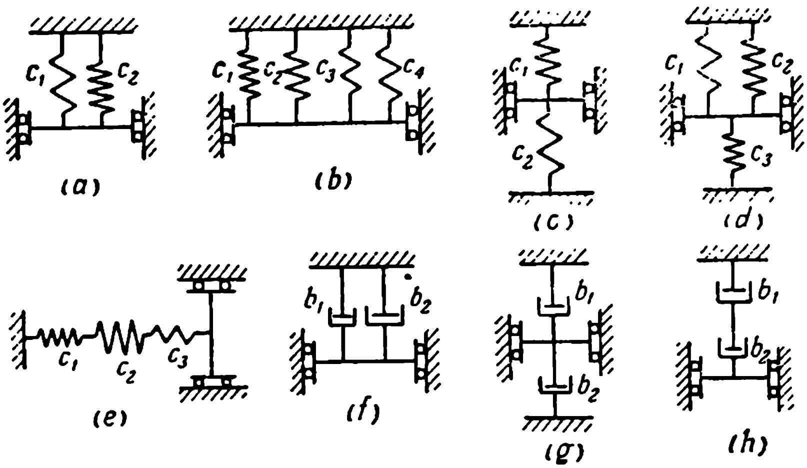


Figure 59

shown in Fig. 59*a-d*. The total stiffness c of the group is equal to the sum of the stiffnesses c_i of its elements

$$c = \sum_{i=1}^{n} c_i \tag{1}$$

where n is the number of the group elements.

A group of elastic elements in series is illustrated in Fig. 59*e*. The total compliance of the group, $1/c$ (*compliance* is the reciprocal of stiffness), is equal to the sum of the compliances $1/c_i$ of its elements:

$$\frac{1}{c} = \sum_{i=1}^{n} \frac{1}{c_i} \tag{2}$$

The coefficient of resistance of a group of damping elements in parallel is defined like the total stiffness (Fig. 59*f* and *g*):

$$b = \sum_{i=1}^{n} b_i \tag{3}$$

The mobility[1] $1/b$ of a group of damping elements in series is defined like the compliance (Fig. 59*h*):

$$\frac{1}{b}=\sum_{i=1}^{n}\frac{1}{b_i} \tag{4}$$

Suppose that lever *1* in the system shown in Fig. 60*a* can perform only small oscillations; one end of the lever is pivoted at *2*. The

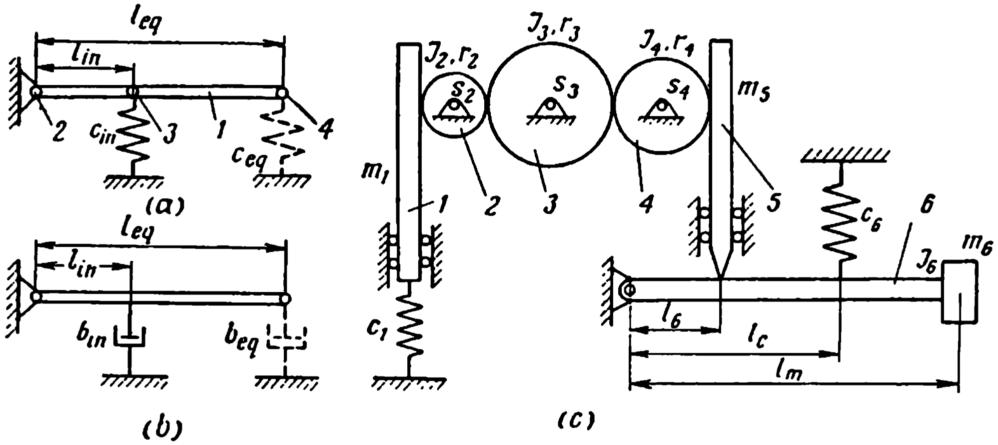


Figure 60

lever is supported by a spring of stiffness c_{in} at point *3* at a distance of l_{in} from the pivot. We are concerned with the motion of the lever end *4* at a distance of l_{eq} from the pivot. The equivalent stiffness c_{eq} of the spring supporting the lever end is to be determined and the action of this spring must be equivalent to that of the spring installed as shown in the diagram.

The criterion of equivalence of the installed and equivalent springs, which provides the equality of the natural frequencies, consists in that the potential energies stored by the springs when the lever is displaced (turned about the pivot) must be equal

$$\frac{c_{in}x_{in}^2}{2}=\frac{c_{eq}x_{eq}^2}{2}$$

where x_{in} = deformation of the installed spring due to lever displacement

x_{eq} = deformation of the equivalent spring with the same lever displacement.

Since

$$\frac{x_{in}}{x_{eq}}=\frac{l_{in}}{l_{eq}}$$

[1] The term *mobility* is analogous to the term *conductance* which is used in electrical engineering. Conductance is reciprocal of electrical resistance.

we obtain

$$c_{eq}=c_{in}\frac{l_{in}^2}{l_{eq}^2} \tag{5}$$

The procedure of finding the equivalent coefficients of resistance (the arrangement is schematically illustrated in Fig. 60*b*) is similar to the above. The criterion of equivalence of the installed and equivalent dampers which provides the equality of the damping coefficients is that the power dissipated by the dampers with a displacement of the lever must be the same

$$b_{in}\dot{x}_{in}^2=b_{eq}\dot{x}_{eq}^2$$

where b_{in} and b_{eq} = resistance coefficients of the installed and equivalent dampers

$\dot{x}_{in}$ and $\dot{x}_{eq}$ = lever velocity at the points of connection of the respective dampers.

Since

$$\frac{\dot{x}_{in}}{\dot{x}_{eq}}=\frac{l_{in}}{l_{eq}}$$

we may write

$$b_{eq}=b_{in}\frac{l_{in}^2}{l_{eq}^2} \tag{6}$$

If a point mass m_{in} installed at point *3* of lever *1* is to be reduced to point *4*, i.e., if we want to find the equivalent mass m_{eq}, then the criterion of equivalence providing the equality of the natural frequencies is that the kinetic energies be equal:

$$\frac{m_{in}\dot{x}_{in}^2}{2}=\frac{m_{eq}\dot{x}_{eq}^2}{2}$$

whence

$$m_{eq}=m_{in}\frac{l_{in}^2}{l_{eq}^2} \tag{7}$$

The angular stiffness, angular coefficient of resistance and moment of inertia are reduced in a similar manner.

Consider, for example, the single-degree-of-freedom system illustrated in Fig. 60*c*. Rack *1*, gears *2*, *3*, and *4* and rack *5* are in mesh. Rack *5* is connected to lever *6*. Gears *2*, *3*, and *4* and the pivot of lever *6* have fixed axes. Rack *1* and lever *6* are connected to springs. Elastic torsional elements are connected to the gear shafts. The designations of the masses m_i, moments of inertia J_i, pitch line radii r_i, angular stiffnesses s_i and stiffnesses c_i of the respective elements as well as of the arms of lever *6* are shown in the figure. It is necessary to reduce all the inertia elements to mass m_1 and all the elastic

elements to spring c_1 (we consider here small oscillations of lever *6*).

Introducing the notations x_i and φ_i for the linear and angular displacements of the respective elements, we can write

$$|x_1| = r_2|\varphi_2| = r_3|\varphi_3| = r_4|\varphi_4| = |x_5| = l_6|\varphi_6| \tag{8}$$

The total kinetic energy of the system is

$$\frac{m_{eq}\dot{x}_1^2}{2} = \frac{m_1\dot{x}_1^2}{2} + \frac{J_2\dot{\varphi}_2^2}{2} + \frac{J_3\dot{\varphi}_3^2}{2} + \frac{J_4\dot{\varphi}_4^2}{2} +$$

$$+ \frac{m_5\dot{x}_5^2}{2} + \frac{J_6\dot{\varphi}_6^2}{2} + \frac{m_6 l_m^2\dot{\varphi}_6^2}{2}$$

Hence, making use of expression (8), we obtain the equivalent mass m_{eq}:

$$m_{eq} = m_1 + m_5 + m_6\frac{l_m^2}{l_6^2} + \frac{J_2}{r_2^2} + \frac{J_3}{r_3^2} + \frac{J_4}{r_4^2} + \frac{J_6}{l_6^2}$$

The total potential energy of the system is

$$\frac{c_{eq}x_1^2}{2} = \frac{c_1x_1^2}{2} + \frac{s_2\varphi_2^2}{2} + \frac{s_3\varphi_3^2}{2} + \frac{s_4\varphi_4^2}{2} + \frac{c_6 l_c^2\varphi_6^2}{2}$$

from which, using expressions (8), we obtain the total stiffness:

$$c_{eq} = c_1 + c_6\frac{l_c^2}{l_6^2} + \frac{s_2}{r_2^2} + \frac{s_3}{r_3^2} + \frac{s_4}{r_4^2}$$

Not infrequently there arises the problem of approximately reducing a multi-degree-of-freedom system or a system with distributed parameters to a system having a smaller number of degrees of freedom, most often with one degree of freedom. Consider, for instance, a system comprising element *1* of mass m suspended from spring *2* of mass m_1 (Fig. 61*a*). The spring mass is uniformly distributed so that the system has an infinite number of degrees of freedom. Let us reduce approximately the spring mass to the spring end connected to element *1*. As a result, a single-degree-of freedom system will be obtained.

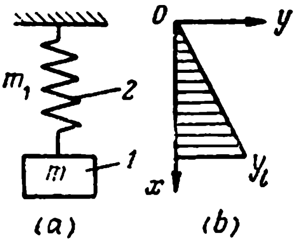


Figure 61

Let us denote by x the coordinate of a spring cross-section measured along the spring axis from the fixed end and by y the displacement of this cross-section from the equilibrium position. Let us assume that the $y(x)$ diagram of the vibrating system at any moment of time is similar to the diagram of the static deformation of the spring

under the action of a force applied to the lower end of the spring. This diagram is a triangle as shown in Fig. 61*b* and

$$y=\frac{y_l}{l}x$$

where l = length of the spring

y_l = displacement of its lower end.

The above assumption is the more accurate the stronger are the inequalities $m_1 < m$, $2l < \lambda$, where λ is the deformation-wave length in the spring; this length is known to be inversely proportional to the vibration frequency.

We shall determine the equivalent mass by applying the above method of energies. The kinetic energy of an infinitesimal portion of the spring

$$dT=\frac{1}{2}\dot{y}^2 dm_1$$

where

$$dm_1=\frac{m_1}{l}dx$$

Hence, upon using the relation $\dot{y}(x)$, which is analogous to $y(x)$, we obtain

$$dT=\frac{m_1\dot{y}_l^2}{2l^3}x^2\,dx$$

The kinetic energy of the whole spring

$$T=\frac{m_1\dot{y}_l^2}{2l^3}\int_0^l x^2\,dx=\frac{m_1\dot{y}_l^2}{6}$$

On the other hand

$$T=\frac{m_{eq}\dot{y}_l^2}{2}$$

where m_{eq} is the equivalent mass of the spring. Hence

$$m_{eq}=\frac{m_1}{3}$$

27. General Equations of Motion of a System with a Centrifugal Vibration Generator

In studying the vibrations of systems excited by a centrifugal vibration generator we shall assume that the unbalances of the generators rotate uniformly. Though, as shown in Chapter VI, this assumption often does not correspond to reality, it enables us to solve,

with sufficient accuracy, a number of simple problems treated by the dynamics of centrifugal vibration machines.

Even with constant angular velocity of rotation of the unbalances the inertia forces exerted by these unbalances on the system to be driven depend themselves on the motion of the system excited, the character of this relation being determined by a number of factors mentioned in Section 25. In this respect the action of centrifugal vibration generators differs from that of uniformly rotating forces

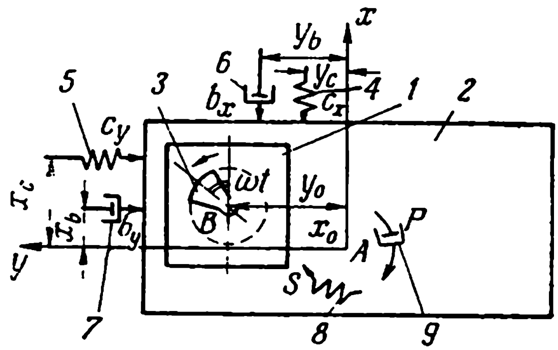


Figure 62

of constant modulus. The former however can be reduced to the latter. In order to elucidate the conditions and methods of reduction it will be useful to study the general case of the motion of a plane system with a centrifugal vibration generator or exciter.

Figure 62 illustrates schematically such an arrangement. The body of vibration generator *1* is rigidly fixed to the driven body *2*. The latter together with the parts of the machine fixed to it is called the *working member*. The centre of gravity of the working member is at point A whose equilibrium position[1] is taken as the origin. Unbalanced mass *3* rotates about the axis passing through point B rigidly fixed to the working member. The numbers *4* and *5* designate the projections of the resultant of elastic forces applied to the working member, and the numbers *6* and *7* are the projections of the resultant of dissipative forces. An elastic moment *8* and a dissipative moment *9* are also applied to the working member.

We introduce the following notations:

m_0 = unbalanced mass

r = eccentricity of the mass relative to the axis of rotation B

x_0, y_0 = projections of point B

$$l = AB = \sqrt{x_0^2 + y_0^2}$$

m_1 = mass of the working member

x_c, y_c = arms of the projections of the elastic force

[1] If stable equilibrium does not exist, the mean position is taken as the origin.

x_b, y_b = arms of the projections of the dissipative force
J_0 = central moment of inertia of the unbalance
J_1 = central moment of inertia of the working member
ω = angular velocity of rotation of the unbalance
x, y = projections of the displacement of point A from the equilibrium position
ψ = angle of rotation of the working member measured from its equilibrium position
t = time.

To derive the differential equations of motion we shall use the Lagrange equations (5), Sec. 10. The system under consideration has three degrees of freedom. We select x, y and ψ as generalized coordinates. We assume the angle ψ to be small and hence x_0 and y_0 are constant. The elastic and dissipative forces are proportional to the corresponding generalized coordinates and velocities. The arms of the moments of elastic and dissipative forces are constant.

The kinetic energy of the system

$$T = T_1 + T_0 \tag{1}$$

where

$$T_1 = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} J_1 \dot{\psi}^2 \tag{2}$$

is the kinetic energy of the working member and

$$T_0 = \frac{1}{2} m_0 (\dot{x} - y_0 \dot{\psi} - \omega r \sin \omega t)^2 + \\ + \frac{1}{2} m_0 (\dot{y} + x_0 \dot{\psi} + \omega r \cos \omega t)^2 + \frac{1}{2} J_0 \omega^2 \tag{3}$$

is the kinetic energy of the unbalance.

The potential energy of the system

$$\Pi = \frac{1}{2} c_x (x - y_c \psi)^2 + \frac{1}{2} c_y (y + x_c \psi)^2 + \frac{1}{2} s \psi^2 \tag{4}$$

where c_x, c_y and s are the coefficients of linear and angular stiffnesses.

The generalized forces

$$\left.\begin{aligned} Q_x &= -b_x (\dot{x} - y_b \dot{\psi}) \\ Q_y &= -b_y (\dot{y} + x_b \dot{\psi}) \\ Q_\psi &= -(p + b_x y_b^2 + b_y x_b^2)\,\dot{\psi} + b_x y_b \dot{x} - b_y x_b \dot{y} \end{aligned}\right\} \tag{5}$$

where b_x, b_y and p are the coefficients of linear and angular resistances.

After the Lagrangian function is determined from formula (6), Sec. 10, and the operations indicated in equations (5), Sec. 10,

are performed, we obtain the required differential equations of vibrations of the system concerned:

$$\left.\begin{aligned} &(m_1+m_0)\ddot{x}+b_x\dot{x}+c_xx-m_0y_0\ddot{\psi}-b_xy_b\dot{\psi}-c_xy_c\psi= \\ &\qquad = m_0r\omega^2\cos\omega t \\ &(m_1+m_0)\ddot{y}+b_y\dot{y}+c_yy+m_0x_0\ddot{\psi}+b_yx_b\dot{\psi}+c_yx_c\psi= \\ &\qquad = m_0r\omega^2\sin\omega t \\ &(J_1+m_0l^2)\ddot{\psi}+(p+b_xy_b^2+b_yx_b^2)\dot{\psi}+ \\ &+(s+c_xy_c^2+c_yx_c^2)\psi-m_0y_0\ddot{x}-b_xy_b\dot{x}-c_xy_cx+ \\ &+m_0x_0\ddot{y}+b_yx_b\dot{y}+c_yx_cy=m_0r\omega^2(x_0\sin\omega t-y_0\cos\omega t) \end{aligned}\right\} \quad (6)$$

If the action of the unbalanced mass is replaced by that of a force of constant modulus F_a applied at point B and rotating at the same angular velocity ω, the differential equations of motion will assume the following form:

$$\left.\begin{aligned} &m_1\ddot{x}+b_x\dot{x}+c_xx-b_xy_b\dot{\psi}-c_xy_c\psi=F_a\cos\omega t \\ &m_1\ddot{y}+b_y\dot{y}+c_yy+b_yx_b\dot{\psi}+c_yx_c\psi=F_a\sin\omega t \\ &J_1\ddot{\psi}+(p+b_xy_b^2+b_yx_b^2)\dot{\psi}+(s+c_xy_c^2+c_yx_c^2)\psi- \\ &-b_xy_b\dot{x}-c_xy_cx+b_yx_b\dot{y}+c_yx_cy=F_a(x_0\sin\omega t-y_0\cos\omega t) \end{aligned}\right\} \quad (7)$$

Comparing expressions (6) and (7), we see that to reduce equations (7) to (6) it is necessary:

(a) to take the modulus of the exciting force F_a equal to the centrifugal force $m_0r\omega^2$ developed by the unbalance in its relative motion about the B axis;

(b) to take as the mass the total mass of the system, i.e., to add the mass m_0 of the rotating unbalance to the mass m_1 of the working member;

(c) to calculate the moment of inertia of the system with respect to the centre of gravity of the working member, taking the unbalanced mass to be localized on the axis of rotation B, i.e., to add the product m_0l^2 to the central moment of inertia J_1 of the working member;

(d) to introduce into the left-hand sides of the differential equations the terms representing the inertia coupling of the rotational degree of freedom with the translational one, the coefficient of the terms determining the coupling between ψ and x being equal to $-m_0y_0$ and the coefficient of the terms determining the coupling of ψ and y to m_0x_0.

If the axis of rotation of the unbalance and the centre of gravity of the working member coincide ($x_0 = y_0 = 0$) or the motion of the working member is translational ($\psi \equiv 0$), then the last two conditions are eliminated.

28. Vibration Generators with a Directed Exciting Force

We shall call the system *centred* if the resultant of the centrifugal forces developed by unbalanced masses, the resultant of the elastic forces applied to the working member and the resultant of dissipative forces applied to it pass all the time through the centre of gravity

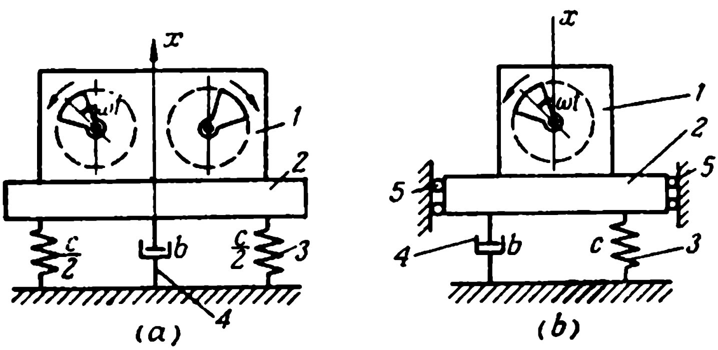


Figure 63

of the working member. A centred system is pictured in Fig. 63*a*; the vibration generator *1* whose action is directed along a straight line is mounted on plate *2* supported by springs *3* and damper *4*; the exciting, elastic and dissipative forces act along the same straight line—the x-axis.

The arrangement in Fig. 63*b*, which shows a circular-action vibration generator, is equivalent to the preceding one provided the idealized constraints *5* warrant a rectilinear motion of the plate along the x-axis; the unbalance and its static mass moment with respect to the axis of rotation are equal to the sum of the masses and to the sum of the mass moments of the preceding arrangement, respectively, and the rest of the parameters of the two systems are identical. The ideal constraints *5* make superfluous the requirement that the forces be centred.

In this case the differential equation of vibrations, in accordance with the first of equations (6), Sec. 27, takes the form

$$(m_1 + m_0)\ddot{x} + b\dot{x} + cx = m_0 r\omega^2 \cos \omega t \tag{1}$$

or

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = \frac{m_0 r\omega^2}{m_1 + m_0} \cos \omega t \tag{2}$$

where

$$h=\frac{b}{2(m_1+m_0)};\quad \omega_0=\sqrt{\frac{c}{m_1+m_0}}\tag{3}$$

The particular integral corresponding to steady-state forced vibrations can be written in the following form:

$$x=x_a\cos(\omega t-\varphi)\tag{4}$$

where [according to formula (3), Sec. 8]

$$x_a=\frac{m_0r\omega^2}{(m_1+m_0)\sqrt{(\omega_0^2-\omega^2)^2+4h^2\omega^2}}$$

and in accordance with formula (21), Sec. 7,

$$\varphi=\tan^{-1}\frac{2h\omega}{\omega_0^2-\omega^2}$$

The amplitude response curves, using the corresponding dimensionless variables (26), Sec. 13, are shown in Figs. 31c and 34a and b.

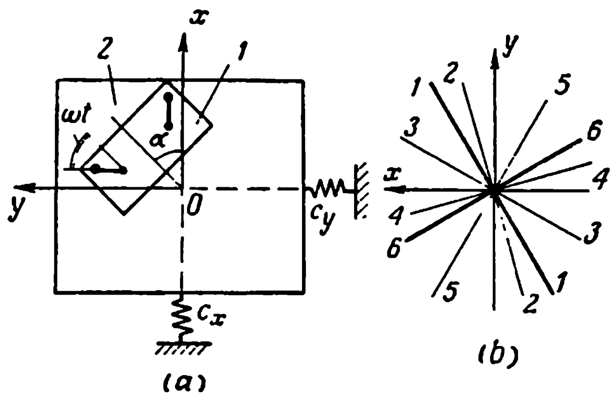


Figure 64

We now turn to the study of the centred system of Fig. 64a in which the line of action of the exciting force coincides with neither of the principal axes of stiffness Ox or Oy[1]. The line of action of the exciting force of the vibration generator *1* is inclined at an angle α to the axis Ox. The origin O is at the centre of gravity of the working member *2* when it is in the equilibrium or mean position.

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[1] In the direction of one of the principal axes the stiffness is maximum, in the direction of the other (at right angles to the former) it is minimum; the direction of the static force coincides with that of the deformation caused by it only if the line of action of the force coincides with one of the principal axes or if the two principal stiffnesses are equal. In the latter case the stiffnesses in any direction are equal and the action of the springs is equivalent to the action of an isotropic elastic medium, possessing no inertia, on the working member of circular section.

Since the system is centred, its vibrations will be only translational. In this case the differential equations may be written as follows:

$$\left.\begin{aligned}(m_1+m_0)\ddot{x}+c_x x=m_0 r\omega^2\cos\alpha\cos\omega t\\(m_1+m_0)\ddot{y}+c_y y=m_0 r\omega^2\sin\alpha\cos\omega t\end{aligned}\right\}\tag{5}$$

or

$$\left.\begin{aligned}\ddot{x}+\omega_x^2 x=\frac{m_0 r\omega^2\cos\alpha}{m_1+m_0}\cos\omega t\\\ddot{y}+\omega_y^2 y=\frac{m_0 r\omega^2\sin\alpha}{m_1+m_0}\cos\omega t\end{aligned}\right\}\tag{6}$$

where

$$\omega_x=\sqrt{\frac{c_x}{m_1+m_0}};\quad \omega_y=\sqrt{\frac{c_y}{m_1+m_0}}\tag{7}$$

The solutions of Eqs. (6) corresponding to steady-state forced vibrations take the form

$$x=\frac{a\cos\alpha}{\omega_x^2-\omega^2}\cos\omega t,\quad y=\frac{a\sin\alpha}{\omega_y^2-\omega^2}\cos\omega t\tag{8}$$

where

$$a=\frac{m_0 r\omega^2}{m_1+m_0}\tag{9}$$

Expressions (8) show that the working member vibrates in a straight line at the amplitude $\sqrt{x_a^2+y_a^2}$. In fact, from these expressions it follows that

$$\tan\alpha_1=\frac{dy}{dx}=\frac{\omega_x^2-\omega^2}{\omega_y^2-\omega^2}\tan\alpha=\text{const}\tag{10}$$

where α_1 is the angle of slope of the path relative to the Ox-axis.

When $\frac{\omega_x^2-\omega^2}{\omega_y^2-\omega^2}>0$, i.e., when the vibrations along the two principal axes of stiffness are simultaneously preresonant or simultaneously postresonant, the paths *2*, *3* of the working member (Fig. 64*b*) lie in the same quadrants as the line of action *1* of the exciting force. If $\omega_x=\omega_y$, i.e., $c_x=c_y$, we obtain from formula (10) $\alpha_1=\alpha$, i.e., the path coincides with the line of action of the force.

When $\frac{\omega_x^2-\omega^2}{\omega_y^2-\omega^2}<0$, i.e., when the vibrations are preresonant along one of the principal axes and postresonant along the other, the paths *4*, *5* of the working member lie in adjacent quadrants. With $\frac{\omega_x^2-\omega^2}{\omega_y^2-\omega^2}=-\cot^2\alpha$ the path *6* of the working member is perpendicular to the line of action of the exciting force.

Figure 65 shows a centred system which differs from the preceding one by the presence of damping, with the principal axes of stiffness Ox and Oy coinciding with the principal axes of damping. The differential equations of motion for this system have the following form:

$$\left.\begin{aligned}(m_1+m_0)\ddot{x}+b_x\dot{x}+c_xx=m_0r\omega^2\cos\alpha\cos\omega t\\(m_1+m_0)\ddot{y}+b_y\dot{y}+c_yy=m_0r\omega^2\sin\alpha\cos\omega t\end{aligned}\right\}\tag{11}$$

or

$$\left.\begin{aligned}\ddot{x}+2h_x\dot{x}+\omega_x^2x=\frac{m_0r\omega^2\cos\alpha}{m_1+m_0}\cos\omega t\\\ddot{y}+2h_y\dot{y}+\omega_y^2y=\frac{m_0r\omega^2\sin\alpha}{m_1+m_0}\cos\omega t\end{aligned}\right\}\tag{12}$$

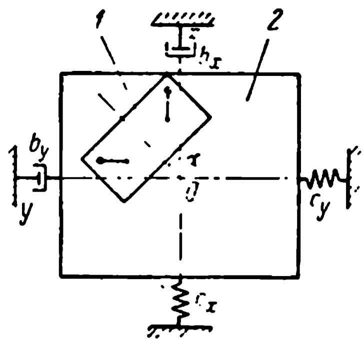


Figure 65

where

$$h_x=\frac{b_x}{2(m_1+m_0)};\quad h_y=\frac{b_y}{2(m_1+m_0)}\tag{13}$$

and ω_x and ω_y are defined by expressions (7).

The solutions of Eqs. (12) corresponding to periodic forced vibrations can be presented in the following form:

$$\left.\begin{aligned}x=x_a\cos(\omega t-\varphi_x)\\y=y_a\cos(\omega t-\varphi_y)\end{aligned}\right\}\tag{14}$$

where

$$\left.\begin{aligned}x_a=\frac{m_0r\omega^2\cos\alpha}{(m_1+m_0)\sqrt{(\omega_x^2-\omega^2)^2+4h_x^2\omega^2}}\\y_a=\frac{m_0r\omega^2\sin\alpha}{(m_1+m_0)\sqrt{(\omega_y^2-\omega^2)^2+4h_y^2\omega^2}}\end{aligned}\right\}\tag{15}$$

$$\varphi_x=\tan^{-1}\frac{2h_x\omega}{\omega_x^2-\omega^2};\quad \varphi_y=\tan^{-1}\frac{2h_y\omega}{\omega_y^2-\omega^2}\tag{16}$$

If $\varphi_x=\varphi_y$, i.e., $\frac{h_x}{h_y}=\frac{\omega_x^2-\omega^2}{\omega_y^2-\omega^2}$, the system behaves like a conservative one, i.e., the working member performs rectilinear vibrations at an amplitude of $\sqrt{x_a^2+y_a^2}$ which is less than the vibration amplitude of a conservative system. The direction of the motion is determined by expression (10) with α_1 and α lying in the same quadrant.

If $\varphi_x\neq\varphi_y$, the working member has a translational elliptical motion; every point moves in its own elliptical path counterclock-

wise at $\varphi_x < \varphi_y$ and clockwise at $\varphi_x > \varphi_y$. One of the axes of the elliptic path is at an angle

$$\alpha_1 = \frac{1}{2}\tan^{-1}\frac{2x_a y_a \cos(\varphi_y - \varphi_x)}{x_a^2 - y_a^2} \tag{17}$$

with respect to the Ox-axis.

The vibration amplitude along this axis

$$a_1 = \frac{x_a y_a \,|\sin(\varphi_y - \varphi_x)|}{\sqrt{\frac{1}{2}(x_a^2 + y_a^2) - \frac{1}{2}(x_a^2 - y_a^2)\cos 2\alpha_1 - x_a y_a \cos(\varphi_y - \varphi_x)\sin 2\alpha_1}} \tag{18}$$

The vibration amplitude along the other axis of the elliptic path

$$b_1 = \frac{x_a y_a \,|\sin(\varphi_y - \varphi_x)|}{\sqrt{\frac{1}{2}(x_a^2 + y_a^2) + \frac{1}{2}(x_a^2 - y_a^2)\cos 2\alpha_1 - x_a y_a \cos(\varphi_y - \varphi_x)\sin 2\alpha_1}} \tag{19}$$

When $x_a = y_a$ and $|\varphi_y - \varphi_x| = \frac{\pi}{2}$, the vibrations performed by the working member are circular.

The study of the system shown in Fig. 66 is of some interest. The working member *1* not subjected to the action of gravity is vibrated by generator *2*; the line of action of the exciting force is at a distance $A_1O = l$ from the centre of gravity O. We select the displacement x of the centre of gravity and the turning angle ψ of the working member from the mean position as generalized coordinates. The differential equations of motion can be written, on the basis of expressions (6), Sec. 27, in the following form:

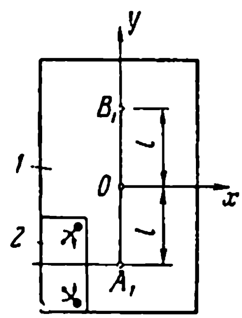


Figure 66

$$\left.\begin{aligned} (m_1 + m_0)\ddot{x} + m_0 l\ddot{\psi} &= m_0 r\omega^2 \cos\omega t \\ (J_1 + m_0 l^2)\ddot{\psi} + m_0 l\ddot{x} &= m_0 r l\omega^2 \cos\omega t \end{aligned}\right\} \tag{20}$$

Inserting the periodic solution of (20)

$$x = A\cos\omega t, \quad \psi = B\cos\omega t \tag{21}$$

and solving for A and B the resulting simultaneous algebraic equations, we obtain

$$A = -\frac{m_0 r J_1}{J_1(m_1 + m_0) + m_1 m_0 l^2}, \quad B = -\frac{m_1 m_0 l r}{J_1(m_1 + m_0) + m_1 m_0 l^2} \tag{22}$$

All the points of the working member on the y-axis vibrate in a horizontal direction and their displacement from the mean position is determined by the expression

$$x_* = (A - By)\cos\omega t \tag{23}$$

where y is the ordinate of the vibrating point.

Let us find the ordinate $y = L$ of such a point B_1 on the Oy-axis that remains motionless at all times, i.e., a point for which $x_* = 0$. From expression (23) we have $A - BL = 0$; hence, using expressions (22), we obtain

$$L = \frac{J_1}{m_1 l} \tag{24}$$

The point B_1 is called the *centre of vibration* or *zero point*. It should be noted that points A_1 and B_1 are reciprocal: if the line of action of the exciting force passes through point B_1, then point A_1 will become the centre of vibration. The zero point or centre of vibration corresponds to the centre of percussion.

Thus, the vibrations of the working member can be considered as swinging motions about the zero point B_1 which itself remains stationary. An ideal hinge can be placed at this point and no reactions will appear at the hinge with small vibrations of the working member.

29. Vibration Generators with a Circular Exciting Force

The working member *1* of the centred system illustrated in Fig. 67 is driven by vibration generator *2* which develops a circular exciting force. Assuming $b_x = b_y = 0$, we can write down the differential equations of the vibrations on the basis of expressions (6), Sec. 27:

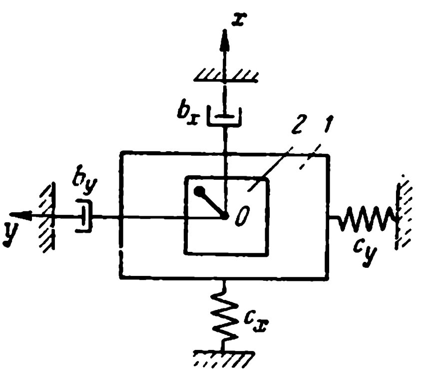


Figure 67

$$\left.\begin{aligned} (m_1+m_0)\ddot{x}+c_x x &= m_0 r\omega^2 \cos \omega t \\ (m_1+m_0)\ddot{y}+c_y y &= m_0 r\omega^2 \sin \omega t \end{aligned}\right\} \tag{1}$$

The equations are not coupled and consequently x and y are the normal coordinates of the system. The solutions of equations (1) describing the forced vibrations take the following form:

$$x = \frac{m_0 r\omega^2}{(m_1+m_0)(\omega_x^2-\omega^2)} \cos \omega t, \quad y = \frac{m_0 r\omega^2}{(m_1+m_0)(\omega_y^2-\omega^2)} \sin \omega t \tag{2}$$

where the natural frequencies ω_x, ω_y are defined by formulas (7), Sec. 28. All the points of the working member describe identical elliptic paths and their equations referred to the centres of the ellipses can be presented in the form

$$\frac{x^2}{x_a^2}+\frac{y^2}{y_a^2} = 1$$

where the ellipse half-axes

$$x_a = \frac{m_0 r\omega^2}{(m_1+m_0)\,|\,\omega_x^2-\omega^2\,|}$$
$$y_a = \frac{m_0 r\omega^2}{(m_1+m_0)\,|\,\omega_y^2-\omega^2\,|} \tag{3}$$

As can be seen from expressions (2), the points of the working member move along their paths in the sense of rotation of the unbalance provided the tuning of the system along both coordinate axes is simultaneously preresonant or simultaneously postresonant. In both cases $\frac{\omega_x^2-\omega^2}{\omega_y^2-\omega^2} > 0$. If the tuning of the system along one of the coordinate axes is preresonant while along the other axis it is postresonant [1], then $\frac{\omega_x^2-\omega^2}{\omega_y^2-\omega^2} < 0$ and the points of the working member move along their paths in a sense opposite to that of rotation of the unbalanced mass.

There are two special cases in which the paths of the points of the working member become circular: if the principal frequencies are equal ($\omega_x = \omega_y \neq \omega$), the motion is in the sense of rotation of the unbalanced mass; if the frequency of the excitation is the rms value of the principal frequencies $\left(\omega = \sqrt{\frac{\omega_x^2+\omega_y^2}{2}}\right)$, the motion is in the opposite sense to that of the unbalanced mass rotation.

The differential equations of vibration of the system shown in Fig. 67 with the coefficients of resistance different from zero can be written as follows:

$$\left.\begin{aligned}(m_1+m_0)\,\ddot{x}+b_x\dot{x}+c_x x &= m_0 r\omega^2\cos\omega t\\ (m_1+m_0)\,\ddot{y}+b_y\dot{y}+c_y y &= m_0 r\omega^2\sin\omega t\end{aligned}\right\} \tag{4}$$

These equations are not coupled. The solutions corresponding to the forced vibrations are determined by the expressions

$$x = x_a\cos(\omega t-\varphi_x),\quad y = y_a\sin(\omega t-\varphi_y) \tag{5}$$

where

$$x_a = \frac{m_0 r\omega^2}{(m_1+m_0)\sqrt{(\omega_x^2-\omega^2)^2+4h_x^2\omega^2}}$$
$$y_a = \frac{m_0 r\omega^2}{(m_1+m_0)\sqrt{(\omega_y^2-\omega^2)^2+4h_y^2\omega^2}} \tag{6}$$

The quantities φ_x and φ_y are defined by formulas (16), Sec. 28; ω_x and ω_y by formulas (7), Sec. 28; and h_x and h_y by formulas (13), Sec. 28.

[1] According to V. Zemskov this tuning is called *interresonant*.

The working member of the system discussed performs elliptical vibrations but the axes of the elliptical path are not, generally speaking, parallel to the coordinate axes. One of the axes of the elliptical path is inclined with respect to the Ox axis at the angle

$$\alpha_1 = \frac{1}{2} \tan^{-1} \frac{2x_a y_a \sin(\varphi_y - \varphi_x)}{y_a^2 - x_a^2} \tag{7}$$

The amplitude of the vibration displacement along this axis

$$a_1 = \frac{x_a y_a \,|\cos(\varphi_y - \varphi_x)|}{\sqrt{\frac{1}{2}(x_a^2 + y_a^2) - \frac{1}{2}(x_a^2 - y_a^2)\cos 2\alpha_1 + x_a y_a \sin(\varphi_y - \varphi_x)\sin 2\alpha_1}} \tag{8}$$

The amplitude of the vibration displacement along the other axis of the elliptical path

$$b_1 = \frac{x_a y_a \,|\cos(\varphi_y - \varphi_x)|}{\sqrt{\frac{1}{2}(x_a^2 + y_a^2) + \frac{1}{2}(x_a^2 - y_a^2)\cos 2\alpha_1 - x_a y_a \sin(\varphi_y - \varphi_x)\sin 2\alpha_1}} \tag{9}$$

When $|\varphi_y - \varphi_x| < \frac{\pi}{2}$, the working member moves in the sense of rotation of the unbalanced mass. If $|\varphi_y - \varphi_x| > \frac{\pi}{2}$, the working member moves in the opposite sense to that of the unbalanced mass rotation. With $|\varphi_y - \varphi_x| = \frac{\pi}{2}$ the elliptical path degenerates into a rectilinear one with an amplitude $a_1 = \sqrt{x_a^2 + y_a^2}$ and an angle $\alpha_1 = \tan^{-1}\left[\frac{y_a}{x_a}\sin(\varphi_y - \varphi_x)\right]$. When $x_a = y_a$ and $\varphi_y - \varphi_x = 0$, the working member performs circular vibrations in the sense of rotation of the unbalanced mass. The necessary and sufficient condition for the realization of the last case is that $h_x = h_y$, $\omega_x = \omega_y$.

If there are no springs in the system shown in Fig. 67, i.e., $c_x = c_y = 0$, then the following values of the displacement amplitudes and phases must be inserted into the solutions (5):

$$x_a = \frac{m_0 r\omega}{(m_1 + m_0)\sqrt{\omega^2 + 4h_x^2}}, \qquad y_a = \frac{m_0 r\omega}{(m_1 + m_0)\sqrt{\omega^2 + 4h_y^2}} \tag{10}$$

$$\varphi_x = \tan^{-1}\left(-\frac{2h_x}{\omega}\right), \qquad \varphi_y = \tan^{-1}\left(-\frac{2h_y}{\omega}\right) \tag{11}$$

Since in this case $|\varphi_y - \varphi_x| < \frac{\pi}{2}$, the working member can perform only an elliptical motion (in a special case, a circular motion) in the direction of rotation of the unbalanced mass.

Having studied the behaviour of centred systems (or equivalent systems that can have only a translational motion because of the restraints imposed), we consider it useful to lay stress on a number of their features, having in mind that some of them may seem, at first sight, to be paradoxical or even unlikely.

Firstly, centred systems can be constructed whose working member will perform the following vibrations when driven by a vibration generator with directed action: along a rectilinear path at any angle to the line of action of the exciting force, including the right angle; in an elliptical path whose axes may be at any angle to the line of action of the exciting force; in a circular path.

Secondly, centred systems can be constructed whose working member will perform the following vibrations when driven by a vibration generator of circular action: along a rectilinear path at any angle to the principal axes of stiffness and resistance; in an elliptical path (in a special case, in a circular path) in the sense of rotation of the exciting-force vector; in an elliptical (in a special case, circular) path in the opposite sense to that of rotation of the exciting-force vector; the axes of the elliptical path in this and the preceding case may be at any angle to the principal axes of stiffness and resistance.

30. Force Interactions and Power Consumption

Having integrated the differential equations of motion, one can easily calculate the reactions of the elastic (spring) and dissipative (damper) connections. The reaction of the ith elastic element

$$S_i = -c_i z_i \tag{1}$$

where c_i = coefficient of stiffness (linear or angular) of the elastic element

z_i = deformation of the elastic element (linear or angular)

S_i = reaction (force or moment) of the elastic element.

The reaction of the ith dissipative element

$$B_i = -b_i \dot{z}_i \tag{2}$$

where b_i = coefficient of resistance (linear or angular) of the dissipative element

$\dot{z}_i$ = rate of deformation of the dissipative element (linear or angular)

B_i = reaction (force or moment) of the dissipative element.

The projections onto the coordinate axes of the force of interaction (pressure force of the unbalance acting on the bearings) between the

unbalance and the working member are determined by the expressions

$$\left.\begin{aligned} X &= m_1\ddot{x} + \sum_{i=1}^{n} S_{ix} + \sum_{i=1}^{k} B_{ix} \\ Y &= m_1\ddot{y} + \sum_{i=1}^{n} S_{iy} + \sum_{i=1}^{k} B_{iy} \end{aligned}\right\} \tag{3}$$

where x, y = running coordinates of the centre of gravity of the working member

$S_{ix}, B_{ix}, S_{iy}, B_{iy}$ = projections onto the respective coordinate axis of the reaction forces in the linear, elastic or dissipative connections

n, k = total number of linear elements, elastic and dissipative, respectively.

There is no need to carry out labour-consuming calculations using the above formulas since by correlating the first two equations (6), Sec. 27, with formulas (3), we can obtain the relations

$$\left.\begin{aligned} X &= m_0 r\omega^2 \cos \omega t - m_0\ddot{x} + m_0 y_0 \ddot{\psi} \\ Y &= m_0 r\omega^2 \sin \omega t - m_0\ddot{y} - m_0 x_0 \ddot{\psi} \end{aligned}\right\} \tag{4}$$

representing the projections of the exciting force of a circular vibration generator. The correlation of Eqs. (5), Sec. 28, with formulas (3) yields the relations

$$\left.\begin{aligned} X &= m_0 r\omega^2 \cos \alpha \cos \omega t - m_0\ddot{x} \\ Y &= m_0 r\omega^2 \sin \alpha \cos \omega t - m_0\ddot{y} \end{aligned}\right\} \tag{5}$$

for a vibration generator with a directed exciting force in a centred system.

The total pressure force of the unbalance acting on the bearings

$$P = \sqrt{X^2 + Y^2} \tag{6}$$

The structure of relations (4) shows that for a vibration generator with a circular exciting force the maximum value of the pressure force of the unbalance acting on the bearings may be either greater or less than the centrifugal force of the unbalance in its relative motion about the axis of rotation.

Special cases are possible in which the pressure force of the unbalance applied to the bearings is identically zero. In a centred system this is realized when x $-r \cos \omega t$, $y = -r \sin \omega t$. In this case $c_x = c_y = m_1\omega^2$, $b_x = b_y = 0$.

For a vibration machine with a directed exciting force the amplitude of the pressure force produced by each of the unbalances and

applied to its bearings cannot be less than the centrifugal force in the motion about the axis of rotation. It should be kept in mind that formulas (5) as well as expression (6) for such a generator furnish the resultant of the pressures of all the unbalances on their bearings. The pressure of each of the unbalances on its bearings can be calculated in each case as the product, taken with the opposite sign, of the unbalance mass and the absolute acceleration of its centre of gravity. For the system shown in Fig. 62 we obtain

$$\left.\begin{aligned} X &= -m_0(\ddot{x} - y_0\ddot{\psi} - r\omega^2\cos\omega t) \\ Y &= -m_0(\ddot{y} + x_0\ddot{\psi} - r\omega^2\sin\omega t) \end{aligned}\right\} \tag{7}$$

For one of the unbalances of the system pictured in Fig. 65 we have

$$\left.\begin{aligned} X_1 &= -\frac{m_0}{2}[\ddot{x} - r\omega^2\cos(\alpha + \omega t)] \\ Y_1 &= -\frac{m_0}{2}[\ddot{y} - r\omega^2\sin(\alpha + \omega t)] \end{aligned}\right\} \tag{8}$$

and for the other unbalance

$$\left.\begin{aligned} X_2 &= -\frac{m_0}{2}[\ddot{x} - r\omega^2\cos(\alpha - \omega t)] \\ Y_2 &= -\frac{m_0}{2}[\ddot{y} - r\omega^2\sin(\alpha - \omega t)] \end{aligned}\right\} \tag{9}$$

Using D'Alembert's principle, we can now write an expression for the moment M applied to the rotor shaft by the driving motor (Fig. 62):

$$M = m_0 r[\ddot{y}\cos\omega t - \ddot{x}\sin\omega t + (x_0\cos\omega t + y_0\sin\omega t)\ddot{\psi}] \tag{10}$$

or

$$M = m_0 r[\ddot{y}\cos\omega t - \ddot{x}\sin\omega t + l\sin(\omega t + \chi_1)\ddot{\psi}] \tag{11}$$

where

$$l = \sqrt{x_0^2 + y_0^2}$$

$$\chi_1 = \tan^{-1}\frac{x_0}{y_0}$$

Since

$$y = y_a\sin(\omega t - \varphi_y), \quad x = x_a\cos(\omega t - \varphi_x)$$

$$\psi = \psi_a\cos(\omega t + \chi_1 - \chi) \tag{12}$$

the moment applied to the rotor shaft in the general case consists of two terms: the constant component M_{mean} and the sinusoidally

varying component $M_a \cos(2\omega t - \theta)$ oscillating at a frequency twice that of the exciting factor. Thus

$$M = M_{mean} + M_a \cos(2\omega t - \theta) \tag{13}$$

Only in the absence of energy dissipation $M_{mean} = 0$. The value of M_{mean} can be calculated from the formula

$$M_{mean} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} M\,dt \tag{14}$$

Since the unbalanced mass rotates at constant angular velocity ω, the power N transmitted by the motor to the vibrating system

$$N = M\omega \tag{15}$$

Hence

$$N_{mean} = M_{mean}\omega, \quad N_a = M_a\omega \tag{16}$$

A number of types of centrifugal vibration generators are in use, which differ in the manner of realizing the force interaction between

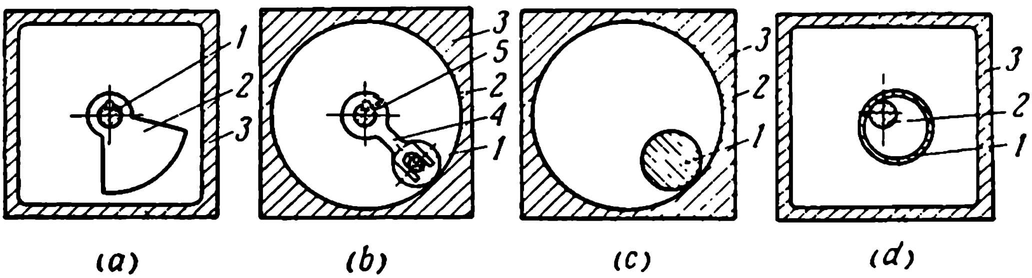


Figure 68

the unbalance and the generator body and in the method of imparting rotational motion to the unbalanced mass. In the above discussion of the force interaction between the unbalance and its shaft bearings we were speaking of the arrangement pictured in Fig. 68*a*. Shaft *1* of the unbalanced mass *2* rotates in ball-bearings whose outer races are rigidly fixed in the body *3* of the vibration machine. This type of machine is generally called *unbalance vibration generator.*

Another type of generator—the carrier planetary type—is schematically illustrated in Fig. 68*b*; in this type runner *1* rolls over runway *2* in the body *3* of the generator. The runner is moved by carrier *4* mounted on the driving shaft *5*. The shaft rotates in ball-bearings whose outer races are fixed in the generator body. If the inertia properties of the shaft and carrier are neglected, the runner may be regarded as an unbalance. The eccentricity r of the runner mass m_0 relative to the axis of rotation is equal to the difference between the radius of the runway R_1 and that of the runner R_2:

$$r = R_1 - R_2 \tag{17}$$

The normal component of the force with which the runner acts on the body is transmitted at the site of contact of the runner and the runway. The tangential component is transmitted to the body via the shaft bearings.

A third type of vibration generator—the friction or gear planetary one—is schematically illustrated in Fig. 68*c*. As in the preceding arrangement, runner *1* rolls over runway *2* in body *3*. The runner is rotated by a shaft that can transmit only the torque. The rolling motion of the runner on the runway is ensured either by the friction force (frictional-planetary type) or by a special gear drive (gear-planetary type). In the former type both the normal and tangential reaction components of the runner are transmitted to the body at the site of contact between runner and runway. In the latter type the tangential component is transmitted by the gearing.

The angular frequency of the excitation generated by the planetary vibrator is equal to the angular velocity ω of the circular motion of the runner over the runway and is related to the angular velocity ω_{sh} of the shaft driving the runner about its axis as follows

$$\omega = i\omega_{sh} \tag{18}$$

where the transmission ratio

$$i = \frac{R_2}{R_1 - R_2} \tag{19}$$

In this case the runner is to be taken as an unbalance; the mass eccentricity is determined by the relation (17). The moment M at the shaft is i times that obtained from formula (13); however, in calculating the power transmitted to the vibrating system the moment calculated by formula (13) should be inserted into formula (14).

Figure 68*d* shows another variety of the planetary vibration generator. In this machine, runner *1* which has the shape of a ring rolls with its inner surface in contact with pin *2* rigidly fixed to body *3*. Since in this case $R_1 < R_2$, the eccentricity of the unbalanced mass is found from the formula

$$r = R_2 - R_1 \tag{20}$$

The transmission ratio is now defined by the expression

$$i = \frac{R_2}{R_2 - R_1} \tag{21}$$

31. Pendulum Vibration Generators

The pendulum vibration generator (or vibrator[1]) shown in Fig. 69 consists of bedplate *1* which is rigidly attached to body *2* to be

[1] The term *vibrator* has long been incorrectly used in the literature as a synonym of a vibration generator. It is well known that in physics, radio engineering, acoustics, and the theory of vibration the word *vibrator* is identified with an *oscillator*, i.e., an *oscillatory system*, and in this sense it does not cover the concept of *forced vibrations*.

vibrated, pendulum *3* swinging about the axis O fixed to the bedplate and unbalanced mass *4* rotating about the axis A fixed to the pendulum. The working member of the centred system (the body vibrated together with the bedplate) performs practically rectilinear vibrations provided the parameters have been suitably chosen. This system will be considered centred if with the swinging pendulum in the mean position the centre of gravity B of the working member, the pivot axis O, the pendulum centre of gravity E and the axis of rotation A of the unbalanced mass are all in one straight line if the lines of action of the resultant of elastic forces and of the dissipative forces applied to the working member by the external medium pass through the centre of gravity B of the working member and if one of the principal axes of stiffness and one of the principal axes of resistance offered by the constraints between the working member and the external medium coincide with the straight line $BOEA$.

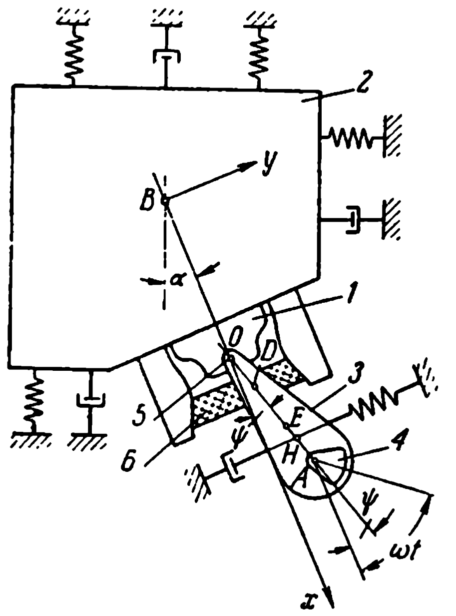


Figure 69

In order to keep the pendulum in its equilibrium position at the required angle to the vertical an elastic bushing *5* or elastic elements *6* are provided for in the design. We introduce the following notations:

m_0 = unbalanced mass
m_1 = mass of pendulum
m_2 = mass of working member
s = coefficient of angular (torsional) stiffness of bushing *5*
p = coefficient of angular resistance of bushing *5*
c = coefficient of stiffness of elements *6*
b = coefficient of resistance of elements *6*
c' and b' = stiffness and resistance coefficients of elastic and dissipative constraints between the pendulum and the external medium
c_x, c_y and s_2 = coefficients of stiffness along the x- and y-axes and the coefficient of angular stiffness of elastic constraints between the working member and the external medium, respectively
b_x, b_y and p_2 = coefficients of resistance along the x- and y-axes

and the coefficient of angular resistance of dissipative constraints between the working member and the external medium, respectively

h = distance BO between the centre of gravity of the working member and the pendulum pivot axis

a = distance OE from the pivot axis of pendulum to its centre of gravity

l = distance OA from the pivot axis of pendulum to the axis of rotation of the unbalanced mass

l_1 = distance OD from the pivot axis of pendulum to the line of action of elastic and dissipative forces of elements *6*

k_1, k = distance OH from the pivot axis of pendulum to the line of action on the pendulum of the elastic and dissipative constraints imposed by the external medium

r = eccentricity of the unbalanced mass relative to the axis of rotation A

J_0 = central moment of inertia of unbalanced mass

J_1' = central moment of inertia of pendulum

J_2' = central moment of inertia of working member

x, y and φ = coordinates of the centre of gravity and the angle of rotation of the working member, measured from the mean position

ψ = angle of rotation of pendulum measured from the mean position

α = angle between the vertical directed downward and the mean position of the line $BOEA$

ω = angular velocity of rotation of unbalanced mass

t = time

ωt = angle between the radius-vector of the centre of gravity of unbalance drawn from the axis of rotation and the positive direction of the x-axis coinciding with the line $BOEA$ in its mean position

g = acceleration of gravity.

In order to find out the conditions under which the working member performs rectilinear translational vibrations along the x-axis we shall set up the general equations of motion assuming the angles φ and ψ to be small:

$$(m_2+m_1+m_0)\ddot{x}+b_x\dot{x}+c_x x=m_0 r\omega^2\cos\omega t \tag{1}$$

$$(m_2+m_1+m_0)\ddot{y}+(b_y+b')\dot{y}+(c_y+c')y+(m_1+m_0)h\ddot{\varphi}+b'h\dot{\varphi}+$$

$$+c'h\varphi+(m_1a+m_0l)\ddot{\psi}+b'k\dot{\psi}+c'k_1\psi=m_0r\omega^2\sin\omega t \tag{2}$$

$$[J_2'+(m_1+m_0)h^2]\ddot{\varphi}+(p_2+b'h^2+p+bl_1^2)\dot{\varphi}+$$

$$+(s_2+c'h+s+cl_1^2)\,\varphi+(m_1a+m_0l)\,\ddot{y}+b'h\dot{y}+c'hy+$$
$$+(m_1a+m_0l)\,h\ddot{\psi}+(b'kh-p-bl_1^2)\,\dot{\psi}+$$
$$+(c'k_1h-s-cl_1^2)\,\psi=m_0r\omega^2h\sin\omega t \tag{3}$$

$$(J_1'+m_1a^2+m_0l^2)\,\ddot{\psi}+(b'k^2+p+bl_1^2)\,\dot{\psi}+$$
$$+(c'k_1^2+m_1ga\cos\alpha+s+cl_1^2)\,\psi+(m_1a+m_0l)\,\ddot{y}+b'k\dot{y}+c'k_1y+$$
$$+(m_1a+m_0l)\,h\ddot{\varphi}+(b'kh-p-bl_1^2)\,\dot{\varphi}+(c'k_1h-s-cl_1^2)\,\varphi=$$
$$=m_0r\omega^2l\sin\omega t \tag{4}$$

The conditions sought are those under which

$$y\equiv0,\quad \varphi\equiv0 \tag{5}$$

We now rewrite Eqs. (2), (3) and (4), bringing their right-hand sides to the same form and making use of identities (5):

$$(m_1a+m_0l)\,\ddot{\psi}+b'k\dot{\psi}+c'k_1\psi=m_0r\omega^2\sin\omega t \tag{6}$$

$$(m_1a+m_0l)\,\ddot{\psi}+\left(b'k-\frac{p+bl_1^2}{h}\right)\dot{\psi}+\left(c'k_1-\frac{s+cl_1^2}{h}\right)\psi=m_0r\omega^2\sin\omega t \tag{7}$$

$$\frac{1}{l}(J_1'+m_1a^2+m_0l^2)\,\ddot{\psi}+\frac{1}{l}(b'k^2+p+bl_1^2)\,\dot{\psi}+$$
$$+\frac{1}{l}(c'\overline{k_1^2}+m_1ga\cos\alpha+s+cl_1^2)\,\psi=m_0r\omega^2\sin\omega t \tag{8}$$

Thus we have obtained three differential equations of the form

$$A_i\ddot{\psi}+B_i\dot{\psi}+C_i\psi=F_a\sin\omega t,\quad (i=1,\,2,\,3) \tag{9}$$

whose particular integrals corresponding to periodic forced vibrations must be identical. To ensure their identity it is necessary and sufficient that the following equalities be satisfied:

$$A_1-\frac{C_1}{\omega^2}=A_2-\frac{C_2}{\omega^2}=A_3-\frac{C_3}{\omega^2} \tag{10}$$

$$B_1=B_2=B_3 \tag{11}$$

Comparing Eqs. (6) and (7) in accordance with expression (10), we obtain

$$\frac{s+cl_1^2}{h}=0$$

whence

$$s=0,\quad c=0 \tag{12}$$

since $l_1^2>0$ and $h\neq0$.

Comparing the same equations according to expression (11), we obtain

$$\frac{p+bl_1^2}{h}=0$$

from which it follows that

$$p=0, \quad b=0 \tag{13}$$

From the comparison of Eqs. (6) and (8) on the basis of expression (11) and taking into account the conditions (13) we find that

$$k=l \tag{14}$$

Finally, comparing Eqs. (6) and (8) by using expression (10) and conditions (12), we obtain

$$l=\frac{J_1}{m_1a}-\frac{1}{\omega^2}\left[\frac{c'k_1(k_1-l)}{m_1a}+g\cos\alpha\right] \tag{15}$$

where the moment of inertia of the pendulum with respect to the axis of rotation

$$J_1=J_1'+m_1a^2 \tag{16}$$

Expressions (12) through (15) are the necessary and sufficient conditions for the identities (5) to be fulfilled. Thus, for the working member in a centred system with a pendulum vibration generator to perform rectilinear vibrations it is necessary and sufficient to observe the following conditions: absence of dissipative and elastic connections of the pendulum with the working member; the line of action of the dissipative reaction of the external medium applied to the pendulum must pass through the axis of rotation of the unbalanced mass; the distance between the axis of swinging of the pendulum and the axis of rotation of the unbalanced mass must conform to expression (15).

The first of the conditions can be satisfied in very few cases but the elastic connections of the pendulum with the working member are made very compliant, with low values of the stiffness coefficients, and the amount of energy dissipated is small. Therefore the effect of such connections may usually be neglected.

The second term on the right-hand side of expression (15) is, as a rule, small as compared with the first and so the distance between the axis of swinging of the pendulum and the axis of rotation of the unbalanced mass may be determined by the formula

$$l=\frac{J_1}{m_1a} \tag{17}$$

i.e., the axis of rotation of the unbalanced mass must be located at the centre of percussion (centre of swinging) of the pendulum.

In the idealized model of the system considered, which is described by the differential equations (1) through (4), two features, among others, of the behaviour of the system have not been taken into account, viz. the mean positions of the working member vibrating along the x-axis and of the swinging pendulum are not the positions of static equilibrium of these elements. The displacement of the mean position of the vibrating working member is determined by the centrifugal inertia force developed by the swinging pendulum; this force has not been taken into account in the above discussion. The running value of the centrifugal force

$$P = (m_1 a + m_0 l)\,\dot{\psi}^2 \tag{18}$$

It can be seen, for example, from differential equation (6) that the oscillations of the pendulum are governed by the law

$$\psi = \psi_a \sin(\omega t - \theta_1) \tag{19}$$

where θ_1 = phase difference of the oscillations of the pendulum with respect to the phase of rotation of the unbalanced mass

ψ_a = angular displacement amplitude of the oscillations of pendulum which can be expressed by the relation

$$\psi_a = \frac{m_0 r}{m_1 a + m_0 l} \tag{20}$$

since the elastic and dissipative constraints imposed on the pendulum are weak.

In general $\psi_a < 0.1$ and so our assumption of smallness of the angle ψ is justified. Making use of relationships (19) and (20), we now rewrite expression (18) as follows

$$P = \frac{(m_0 r)^2 \omega^2}{m_1 a + m_0 l} \cos^2(\omega t - \theta_1)$$

or

$$P = P_{mean} + P_{mean} \cos 2(\omega t - \theta_1) \tag{21}$$

where the mean value of the centrifugal force

$$P_{mean} = \frac{(m_0 r)^2 \omega^2}{2(m_1 a + m_0 l)} = \frac{1}{2}\psi_a m_0 r \omega^2 \tag{22}$$

This force will cause the mean position of the working member to be displaced by

$$x_{mean} = \frac{P_{mean}}{c_x} = \frac{(m_0 r)^2 \omega^2}{2(m_1 a + m_0 l)\,c_x} = \frac{\psi_a m_0 r \omega^2}{2c_x} \tag{23}$$

The second term on the right-hand side of expression (21) shows that under the action of the centrifugal force of the pendulum the working member will perform additional oscillations at double the frequency of rotation of the unbalanced mass. The amplitude

of these oscillations with the weak elastic and dissipative constraints imposed on the working member can be determined from the following expression:

$$x_{2a}=\frac{P_{mean}}{(m_2+m_1+m_0)\,\omega^2}=\frac{(m_0r)^2}{2\,(m_1a+m_0l)\,(m_2+m_1+m_0)} \tag{24}$$

With the same conditions the amplitude of the fundamental tone of vibrations is, according to differential equation (1),

$$x_a=\frac{m_0r}{m_2+m_1+m_0} \tag{25}$$

Comparing the last two equalities, we obtain

$$x_{2a}=\frac{1}{2}\psi_a x_a \tag{26}$$

that is, the amplitude of the second harmonic is usually small as compared with that of the fundamental tone of vibration of the working member. But the displacement of the mean position of the working member defined by relation (23) may be large enough for a small stiffness coefficient c_x.

Before proceeding to the calculation of the displacement of the mean position of the pendulum we turn now to the power balance of the given system. The mechanical power N developed by the motor on the shaft carrying the unbalance can be represented by the following sum:

$$N=N_0+N_1'+N_1''+N_2 \tag{27}$$

where N_0 = power required to overcome the dissipative resistances to the rotation of the unbalanced mass

N_1' = power required to overcome the friction in the pendulum pivot

N_1'' = power required to overcome the rest of the dissipative resistances to the swinging of pendulum

N_2 = power required to overcome the dissipative resistances to the vibration of the working member.

Assuming that all the resistances to the rotation of the unbalanced mass have been reduced to an equivalent Coulomb friction in the bearings and neglecting the relatively small influence of the swinging of the pendulum and of the vibration of the working member on the pressure in the bearings, we may write

$$N_0=f_0r_0m_0r\omega^3 \tag{28}$$

where f_0 = equivalent coefficient of friction in the bearing of the unbalance

r_0 = equivalent radius of the journal in the bearing of the unbalance.

If conditions (12) through (15) are satisfied, the reaction of the pendulum pivot is constantly directed along the x-axis. Therefore, the power required to overcome the friction in the pivot bearing of the pendulum is defined by the relation

$$N_1' = f_1 r_1 m_0 r \omega^2 \,|\, \dot{\psi} \cos \omega t \,| \,.$$

where f_1 = equivalent coefficient of friction in the pivot bearing of the pendulum
r_1 = equivalent radius of the journal in the pivot bearing of the pendulum.

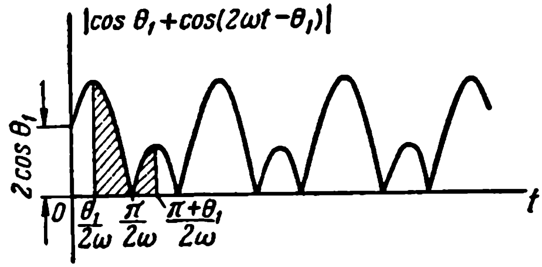


Figure 70

Inserting $\dot{\psi}$ from expressions (19) and (20) into the above relation, we get

$$N_1' = f_1 r_1 \psi_a m_0 r \omega^3 \,|\cos \omega t \cos (\omega t - \theta_1)\,| \tag{29}$$

The reaction component in the pèndulum pivot due to the centrifugal force of the pendulum has not been taken into account in the last expression since this force is much less than the centrifugal force developed by the unbalanced mass.

Upon using an identity transformation expression (29) takes the following more convenient form:

$$N_1' = \frac{1}{2} f_1 r_1 \psi_a m_0 r \omega^3 \,|\cos \theta_1 + \cos (2\omega t - \theta_1)\,| \tag{30}$$

Figure 70 shows a graph representing the dependence of the variable factor in (30) on time. The mean power $N'_{1\,mean}$ can be calculated from the formula

$$N'_{1\,mean} = \frac{2\omega}{\pi} \left(\int\limits_{\frac{\theta_1}{2\omega}}^{\frac{\pi}{2\omega}} N_1'\, dt + \int\limits_{\frac{\pi}{2\omega}}^{\frac{\pi+\theta_1}{2\omega}} N_1'\, dt \right) \tag{31}$$

This quantity is proportional to the shaded area in Fig. 70.

Substituting the value of instantaneous power from equality (30) into expression (31) and integrating, we obtain

$$N'_{1\,mean} = \frac{1}{2} f_1 r_1 \psi_a m_0 r \omega^3 \left[\left(1 - \frac{2\theta_1}{\pi}\right) \cos\theta_1 + \frac{4}{\pi} \sin\theta_1 \right] \tag{32}$$

or, upon using expression (20),

$$N'_{1\,mean} = \frac{f_1 r_1 (m_0 r)^2 \omega^3}{2(m_1 a + m_0 l)} \left[\left(1 - \frac{2\theta_1}{\pi}\right) \cos\theta_1 + \frac{4}{\pi} \sin\theta_1 \right] \tag{33}$$

This expression shows that the mean power $N'_{1\,mean}$ depends on θ_1 which is the phase difference between the pendulum swing and the rotation of the unbalanced mass. However the variation of $N'_{1\,mean}$ over the entire range of θ_1 angles is moderate. The maximum of $N'_{1\,mean}$ attained at $\theta_1 = \pi/2$

$$\max N'_{1\,mean} = \frac{2 f_1 r_1 (m_0 r)^2 \omega^3}{\pi (m_1 a + m_0 l)} \tag{34}$$

The minimum value of $N'_{1\,mean}$ is obtained at $\theta_1 = 0$ and $\theta_1 = \pi$:

$$\min N'_{1\,mean} = \frac{f_1 r_1 (m_0 r)^2 \omega^3}{2(m_1 a + m_0 l)} \tag{35}$$

The ratio $\frac{\min N'_{1\,mean}}{\max N'_{1\,mean}} = 0.785$. Therefore, the error in calculating $N'_{1\,mean}$ due to uncertainty in θ_1 cannot exceed 21.5% of max $N'_{1\,mean}$. In practice the value of this power must approach min $N'_{1\,mean}$ because a large dissipation of energy on swinging of the pendulum should not be tolerated.

The mean power $N''_{1\,mean}$ is determined from formula (4), Sec. 23, which takes in this case the form

$$N''_{1\,mean} = \frac{(m_0 r)^2 l^2 \omega^6 p''_1}{2\{[s_1 - (m_1 a + m_0 l) l\omega^2]^2 + p_1^2 \omega^2\}} \tag{36}$$

where p''_1 = reduced coefficient of angular resistance to the swinging of the pendulum with no allowance for the friction in the pivot bearing

p_1 = ditto, with allowance for the friction in the pivot bearing

s_1 = reduced coefficient of angular stiffness taking account of all elastic constraints imposed on the pendulum.

Since the elastic and dissipative constraints imposed on the pendulum are but slight, expression (36) can be simplified to the following form:

$$N''_{1\,mean} = \frac{(m_0 r)^2 \omega^2 p''_1}{2(m_1 a + m_0 l)^2} \tag{37}$$

If the angle θ_1 is to be calculated, one may use formula (5), Sec. 23, according to which, taking into account that the elastic

constraints imposed on the pendulum are weak, we have

$$N'_{1\,mean}+N''_{1\,mean}=-\frac{(m_0r)^2\,l\omega^3\sin 2\theta_1}{4\,(m_1a+m_0l)} \tag{38}$$

Substituting the values of the mean powers from expressions (33) and (37) into the left-hand side of (38), we obtain the following transcendental equation for calculating the angle θ_1:

$$f_1r_1\left[\left(1-\frac{2\theta_1}{\pi}\right)\cos\theta_1+\frac{4}{\pi}\sin\theta_1\right]+\frac{1}{2}\,l\sin 2\theta_1=-\frac{p''_1}{(m_1a+m_0l)\,\omega} \tag{39}$$

The mean power necessary to sustain the vibration of the working member will be calculated from differential equation (1) on the basis of formula (4), Sec. 23:

$$N_{2\,mean}=\frac{(m_0r)^2\,\omega^6b_x}{2\,\{[c_x-(m_2+m_1+m_0)\,\omega^2]^2+b_x^2\omega^2\}} \tag{40}$$

In accordance with equality (27) the mean mechanical power developed by the motor that drives the shaft of the unbalance is

$$N_{mean}=N_0+N'_{1\,mean}+N''_{1\,mean}+N_{2\,mean} \tag{41}$$

The angular displacement of the mean position of the pendulum from its position of equilibrium can now be calculated. If the motor is built into the pendulum vibration generator, then the mean position is shifted in the opposite sense to that of rotation of the unbalanced mass by

$$\psi_{mean}=\frac{N_{mean}-N_0}{\omega\,[s\pm(m_1a+m_0l)\cos\alpha]} \tag{42}$$

With an external motor the mean position is shifted in the sense of the rotation of the unbalance and

$$\psi_{mean}=\frac{N_0}{\omega\,[s_1\pm(m_1a+m_0l)\cos\alpha]} \tag{43}$$

It has been assumed in both cases that the angle ψ_{mean} is small. The following transcendental equations yield more accurate values for ψ_{mean}:

$$\omega\psi_{mean}\left[s_1\pm(m_1a+m_0l)\,g\sin\frac{\psi_{mean}}{2}\cos\left(\alpha\pm\frac{\psi_{mean}}{2}\right)\right]=N_{mean}-N_0$$

in the first case and

$$\omega\psi_{mean}\left[s_1\pm(m_1a+m_0l)\,g\sin\frac{\psi_{mean}}{2}\cos\left(\alpha\pm\frac{\psi_{mean}}{2}\right)\right]=N_0$$

in the second case.

The plus sign is taken when the centre of gravity of the pendulum in the mean position is higher than in the equilibrium position; otherwise, the minus sign is taken.

The displacement is determined in the first case, from formula (42), by the difference between the moment at the shaft of the unbalance transmitted to the pendulum by the stator and the moment of the resistance to the rotation of the unbalance, the latter moment being transmitted to the pendulum mainly via the bearings. In the second case the displacement is determined from formula (43) by the moment of the forces of resistance to the rotation of the unbalanced mass.

32 Internal Vibration Machines

Internal (poker or immersion) vibration machines are used mainly for compacting concrete mixes. They are centrifugal vibration generators of circular action, of either the unbalance or planetary type. Figure 71 is a schematic of the unbalance type of internal vibration machine whose shaft *2* rotated by an external motor is supported by bearings *3* mounted in the cylindrical case *4*. Unbalanced mass *5* is fixed on the shaft. Shaft *2* is connected with the motor by flexible shaft *1*. At ordinary vibration frequencies the concrete mix may be considered to be an isotropic medium of low stiffness but developing considerable dissipative and inertia forces.

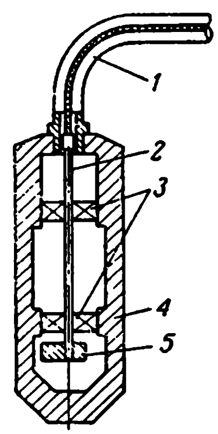


Figure 71

The body of the vibration machine is held so as to prevent its rotation about its axis. Because of the isotropic nature of the medium, the concentricity of the unbalance shaft and the machine body, the cylindrical shape of the body and the axial symmetry of the body masses (including all the elements rigidly fixed to it) the points on the body axis describe circular paths.

Consider the case when the vectors of all the forces applied to the body of the machine (inertia forces of the unbalance and the concrete mix vibrating together with the body, dissipative resistance forces of the concrete mix) are in the same cross-section as the centre of gravity of the body and their lines of action pass through this centre (which, by reason of the axial symmetry of the masses, is on the geometric axis of the body coinciding with the axis of rotation of the unbalanced mass). In this case the body will perform a translational circular motion in which each generatrix of the body will describe a circular cylindrical surface of equal diameter. Hence we may consider the motion in the plane of action of the forces.

In this treatment the arrangement of the internal generator will differ from the one shown in Fig. 67 by the absence of elastic constraints. Besides, in order to take account of the inertia forces developed by the vibrating masses of the concrete mix, the mass m_c of its

co-vibrating part reduced to the body must be added to the mass m_1 of the working member (the machine body). Since the problem under consideration has axial symmetry, the polar coordinates ρ and θ are an adequate system of coordinates to treat it. We place the pole at point O (Fig. 72) about which the axis of the machine body rotates and direct the polar axis OP parallel to the initial position (at time $t = 0$) of the line connecting the body axis A with the centre of gravity B of the unbalance. The direction of the

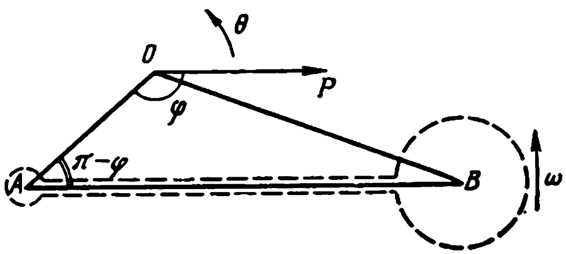


Figure 72

angular velocity ω of the unbalance is counterclockwise; the polar angle θ will also be measured counterclockwise.

The kinetic energy of the system can be represented by the expression

$$T = \frac{1}{2}(m_1 + m_c + m_0)(\dot{\rho}^2 + \rho^2\dot{\theta}^2) +$$

$$+ m_0 r\omega [\rho\dot{\theta}\cos(\omega t - \theta) - \dot{\rho}\sin(\omega t - \theta)] + \frac{1}{2}(J_0 + m_0 r^2)\omega^2 \qquad (1)$$

where $\rho = OA$ — modulus of the radius-vector of the body axis
θ = polar angle of this radius-vector
m_0 = unbalanced mass (including all elements rigidly connected to it)
r = eccentricity of the unbalanced mass relative to the axis of rotation A
J_0 = central moment of inertia of the unbalance.

The dissipative function

$$\Phi = \frac{1}{2} b (\dot{\rho}^2 + \rho^2\dot{\theta}^2) \qquad (2)$$

where b is the coefficient of dissipative resistance of the medium.

Using Lagrange's equations in the form (22), Sec. 10 (putting $L = T$, $Q_i = 0$), we obtain the following system of equations of motion of the internal vibrator:

$$\left.\begin{aligned} &(m_1 + m_c + m_0)(\ddot{\rho} - \rho\dot{\theta}^2) + b\dot{\rho} - m_0 r\omega^2\cos(\omega t - \theta) = 0 \\ &(m_1 + m_c + m_0)(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}) + b\rho\dot{\theta} - m_0 r\omega^2\sin(\omega t - \theta) = 0 \end{aligned}\right\} \qquad (3)$$

It can readily be seen that the first integrals of this nonlinear set of differential equations corresponding to the stationary motion are

$$\dot{\rho}=0, \quad \dot{\theta}=\omega \tag{4}$$

Hence the second integration yields

$$\rho=\rho_a, \quad \theta=\omega t-\varphi \tag{5}$$

where ρ_a = amplitude of the circular vibrations of the machine body

φ = difference between the body phase and the rotation phase of the unbalanced mass ($-\varphi$ is the initial phase angle of the radius-vector ρ).

Substituting the values (4) and (5) in differential equations (3), we obtain the simultaneous equations

$$\left.\begin{aligned}(m_1+m_c+m_0)\,\rho_a+m_0r\cos\varphi=0\\ b\rho_a-m_0r\omega\sin\varphi=0\end{aligned}\right\} \tag{6}$$

the solution of which is

$$\tan\varphi=-\frac{b}{(m_1+m_c+m_0)\,\omega} \tag{7}$$

$$\rho_a=-\frac{m_0r}{m_1+m_c+m_0}\cos\varphi=\frac{m_0r\omega}{b}\sin\varphi=\frac{m_0r\omega}{\sqrt{(m_1+m_c+m_0)^2\,\omega^2+b^2}} \tag{8}$$

As seen from equations (6), $\sin\varphi\geqslant 0$ (the equality sign holds at $b=0$) and $\cos\varphi\leqslant 0$ (the equality sign holds at $\rho_a=0$, i.e., at $b=\infty$). Consequently the angle φ is within the limits $\frac{\pi}{2}\leqslant\varphi\leqslant\pi$, the limit on the right corresponding to $b=0$ and that on the left to $b=\infty$.

The coefficient of dissipative resistance b and the reduced mass of the co-vibrating concrete mix m_c are as a rule unknown. One can measure the amplitude of the circular vibrations of the body ρ_a and the phase difference φ experimentally. Having found these values, we obtain from the second of equations (6):

$$b=\frac{m_0r\omega}{\rho_a}\sin\varphi \tag{9}$$

and from the first of equalities (8)

$$m_c=-\left(\frac{m_0r}{\rho_a}\cos\varphi+m_1+m_0\right) \tag{10}$$

The mean power required to sustain the vibrations of the body

$$N_1=-Bv$$

where B = dissipative force exerted by the medium on the machine body

v = modulus of the velocity of the machine body.

In accordance with expressions (2), (4) and (5)

$$-B = b\rho_a\omega, \quad v = \rho_a\omega$$

Hence

$$N_1 = b\rho_a^2\omega^2$$

Inserting the values of b from (9) and of ρ_a from the first of equalities (8), we obtain

$$N_1 = -\frac{(m_0 r)^2 \omega^3 \sin 2\varphi}{2(m_1 + m_c + m_0)} \tag{11}$$

The expression reaches its maximum value at $\sin 2\varphi = -1$, i.e., at $\varphi = 3\pi/4$. The maximum power

$$\max N_1 = \frac{(m_0 r)^2 \omega^3}{2(m_1 + m_c + m_0)} \tag{12}$$

and is twice that determined from formula (1), Sec. 24. This was to be expected since formula (1), Sec. 24, had been obtained for a single-degree-of-freedom system and in the case considered two degrees of freedom are realized.

The power required to overcome the friction in the bearings of the unbalance shaft

$$N_0 = f_0 r_0 m_0 r \omega^3 \sqrt{1 - \mu(2-\mu)\cos^2\varphi} \tag{13}$$

where f_0 = equivalent coefficient of friction in the bearings
r_0 = equivalent radius of the journal in the bearings

$$\mu = \frac{m_0}{m_1 + m_c + m_0} \tag{14}$$

The relation (13) can be derived by considering the triangle OAB in Fig. 72.

The total mechanical power that can be realized on the unbalance shaft is given by

$$N = N_0 + N_1$$

With an external motor the moment applied to the machine body

$$M = \frac{N_0}{\omega} \tag{15}$$

and is directed in the sense of rotation of the unbalanced mass.

With a built-in motor the moment applied to the body

$$M = \frac{N_1 - N_0}{\omega} \tag{16}$$

and its direction is opposite to the sense of rotation of the unbalanced mass.

In order to prevent the turning of the body of the vibration generator an external moment must be applied to it, the absolute value of which is determined by expression (15) or (16). This statement

is correct in the case when the line of action of the dissipative force intersects the axis of the generator body.

Because of the axial symmetry expressions (6) through (16) also hold at any fixed value of ω in the case of nonlinear dependence of the dissipative force on the velocity and of the inertia force of the concrete mix on the acceleration of motion of the body. In order to plot the amplitude response curve using expression (8) and the phase response curve using (7) it is necessary to know the relations $b = b(\omega)$ and $m_c = m_c(\omega)$.

The plane-parallel motion of the body of the internal generator is in general unacceptable since the fixation point of its handle must vibrate at a sufficiently small (in the ideal case, zero) amplitude. The point on the body axis whose vibration amplitude is zero is called the *zero point*. In order to realize the zero point the centre of gravity of the unbalance is placed below the centre of gravity of the vibration machine. In this case for the motion of the generator body we have four degrees of freedom provided that the body is fixed so that its rotation about its geometric axis and displacement along this axis is prevented.

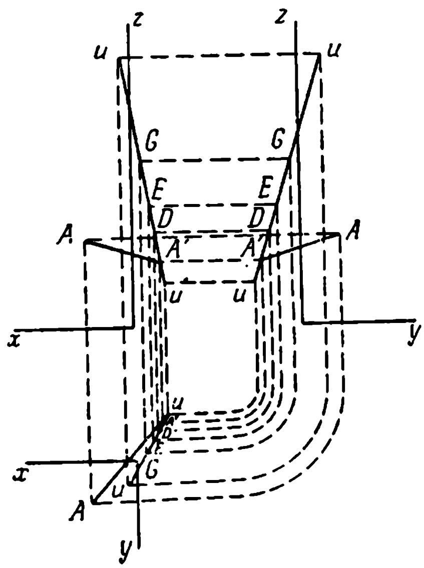


Figure 73

With such an arrangement the motion of the generator body will also possess axial symmetry. All points on its geometric axis will describe circles about a common motionless axis. The centre of gravity of the unbalance will describe a circle about the same axis. Consider a fixed system of cartesian coordinates in which the z-axis has been brought to coincide with the motionless axis mentioned. The x- and y-axes may be chosen arbitrarily. Figure 73 shows the projection of the axis uu of the generator body on the zx-plane (shown on the upper left part); this axis is also the axis of rotation of the unbalanced mass. The centre of gravity of the unbalanced mass is at point A. Point A' is the projection of A onto the u-axis. Assuming the angle between the z- and u-axes and the eccentricity of the unbalanced mass $r = A'A$ to be small compared with the longitudinal dimensions of the vibration generator, we shall take the centre of gravity of the machine to be at point E on the u-axis (with account taken of the co-vibrating mass of the concrete mix reduced to the generator axis). The resultant of the dissipative resistance forces of the concrete mix is applied at point D.

The projections of the same configuration onto the zy-plane and onto the xy-plane are shown on the right and in the lower part of Fig. 73, respectively. The three projections of each of the points mentioned are designated by the same letters. We select as generalized coordinates the abscissa x and the ordinate y of point E of the centre of gravity of the machine and the angles ψ and ε between the z-axis and the projections of the u-axis on the zx- and zy-planes, respectively. We denote by $n = EA'$ the distance from the centre of gravity of the generator to the projection of the centre of gravity of the unbalance on the u-axis; by $n' = ED$ the distance from the centre of gravity of the generator to the point at which the dissipative force is applied and by $n'' = EG$ the distance from the centre of gravity of the generator as a whole to that of the body, the co-vibrating concrete mass being taken into account.

The angles ψ and ε being small, we may write the following expressions of the kinetic energy and dissipative function of our system:

$$T = \frac{1}{2} J_1 (\dot{\psi}^2 + \dot{\varepsilon}^2) + \frac{1}{2} (m_1 + m_c) [(\dot{x} - n''\dot{\psi})^2 + (y - n''\varepsilon)^2] +$$

$$+ \frac{1}{2} m_0 [(\dot{x} + n\dot{\psi} - r\omega \sin \omega t)^2 + (\dot{y} + n\dot{\varepsilon} + r\omega \cos \omega t)^2] \quad (17)$$

$$\Phi = \frac{1}{2} b [(\dot{x} + n'\dot{\psi})^2 + (\dot{y} + n'\dot{\varepsilon})^2] \quad (18)$$

where J_1 is the central moment of inertia of the vibration generator with unbalanced mass with respect to the axis at right angles to the u-axis.

We can now write the differential equations of motion of the body of the vibration generator:

$$(m_1 + m_c + m_0) \ddot{x} + b (\dot{x} + n'\dot{\psi}) = m_0 r\omega^2 \cos \omega t \quad (19)$$

$$J_1 \ddot{\psi} + bn' (\dot{x} + n'\dot{\psi}) = nm_0 r\omega^2 \cos \omega t \quad (20)$$

$$(m_1 + m_c + m_0) \ddot{y} + b (\dot{y} + n'\dot{\varepsilon}) = m_0 r\omega^2 \sin \omega t \quad (21)$$

$$J_1 \ddot{\varepsilon} + bn' (\dot{y} + n\dot{\varepsilon}) = nm_0 r\omega^2 \sin \omega t \quad (22)$$

The equations can be grouped into two subsystems: (19), (20) and (21), (22). The left-hand sides of Eqs. (19) and (21) are analogous as well as the left-hand sides of Eqs. (20) and (22). The right-hand sides of these two pairs of equations differ only in phase by an angle of $\pi/2$ equal to the angle between the x- and y-axes. This is natural in view of the axial symmetry of the problem. It is therefore sufficient to consider the first subsystem.

The condition for the existence of the zero point on the u-axis is that the oscillations of x and ψ be in phase, i.e., their proportionality, and accordingly

$$x = a\psi \quad (23)$$

where the coefficient a is the distance from the centre of gravity E of the machine as a whole to the zero point. Substituting the value of ψ from (23) into Eqs. (19) and (20) and dividing the latter by n, we obtain:

$$\left.\begin{aligned} (m_1+m_c+m_0)\,\ddot{x}+b\left(1+\frac{n'}{a}\right)\dot{x}&=m_0r\omega^2\cos\omega t \\ \frac{J_1}{na}\ddot{x}+b\frac{n'}{n}\left(1+\frac{n'}{a}\right)\dot{x}&=m_0r\omega^2\cos\omega t \end{aligned}\right\} \tag{24}$$

For the integrals of the first and the second equation to be equal it is necessary and sufficient that the coefficients of $\dot{x}$ and $\ddot{x}$ on the left-hand sides of the two equations be equal, respectively. The first of the equalities leads to the relation

$$n'=n \tag{25}$$

and the second yields the expression

$$n=\frac{J_1}{(m_1+m_c+m_0)\,a} \tag{26}$$

Relation (25) indicates that the realization of the zero point requires that the resultant of the dissipative resistance forces be applied to the body axis in the same cross-section in which the unbalance has its centre of gravity. Relation (26) is the second condition for the realization of the zero point. This relation is easily reduced to the form

$$l=\frac{J}{(m_1+m_c+m_0)\,a} \tag{27}$$

where $l=n+a=$ distance from the cross-section of the vibration machine in which the unbalance has its centre of gravity to the zero point

$J=J_1+(m_1+m_c+m_0)\,a^2=$ moment of inertia of the vibration machine with respect to the zero point.

Expression (27) is similar to equality (17), Sec. 31, for the pendulum vibration generator. In fact, if an ideal spherical hinge is placed at the zero point, the hinge will experience no forces acting at right angles to the generator axis.

Denoting the distance from the zero point to the point at which the dissipative force is applied by

$$k=a+n' \tag{28}$$

we may rewrite the first of Eqs. (24) as follows:

$$(m_1+m_c+m_0)\,\ddot{x}+b\frac{k}{a}\dot{x}=m_0r\omega^2\cos\omega t$$

Its solution corresponding to a stationary motion is

$$x = x_a \cos(\omega t - \varphi_1) \tag{29}$$

where

$$\tan\varphi_1 = -\frac{kb}{a(m_1 + m_c + m_0)\,\omega} \tag{30}$$

$$x_a = -\frac{m_0 r}{m_1 + m_c + m_0}\cos\varphi_1 = \frac{a m_0 r\omega}{kb}\sin\varphi_1 =$$

$$= \frac{a m_0 r\omega}{\sqrt{a^2(m_1 + m_c + m_0)^2\,\omega^2 + k^2 b^2}} \tag{31}$$

The last two expressions are somewhat different from equalities (7) and (8) obtained for the plane-parallel motion.

On the basis of what has been said above one can easily derive an expression for the power needed to sustain the vibrations if it is taken into account that $y_a = x_a$, $\varepsilon_a = \psi_a$ and that y and ε lag behind x and ψ by a phase angle of $\pi/2$:

$$N_1 = \frac{bk^2 x_a^2 \omega^2}{a^2}$$

whence, according to expression (31),

$$N_1 = -\frac{k(m_0 r)^2\,\omega^3}{2a(m_1 + m_c + m_0)}\sin 2\varphi_1 \tag{32}$$

With a zero point the axis of the generator body describes the surface of a circular cone. If there is no zero point, the axis describes a ruled surface—a hyperboloid of revolution of one sheet. In this case the power required to sustain the vibrations will be

$$N_1 = b(x_a + n'\psi_a)^2\,\omega^2 \tag{33}$$

where the subsystem of differential equations (19) and (20) must be integrated to determine x_a, ψ_a and the corresponding phase angles φ_1 and φ_2.

CHAPTER 6

NONLINEAR PROBLEMS OF THE DYNAMICS OF CENTRIFUGAL VIBRATION GENERATORS

33. Nonuniformity of Rotation of Unbalances

The problems of the dynamics of centrifugal vibration generators were treated in Chap. 5 on the assumption that their unbalances rotate at a constant angular velocity ω. This led to the description of the motion with the aid of nonautonomous differential equations. However, centrifugal vibration machines are as a rule autonomous systems since the rotation of unbalances is not externally controlled by rigid constraints which ensure a given law of variation of the angular velocity with time, in particular, the uniformity of the velocity. In actual fact, the angular velocity of rotation of unbalances very often does not remain constant.

The variations of the angular velocity of unbalances are small, in general, but there are important problems of the dynamics of centrifugal vibration generators that cannot be solved in principle without taking into account the degrees of freedom corresponding to the rotation of unbalances. In Chapter 5 these additional degrees of freedom were eliminated by superposing the conditions of the constancy of the angular velocity of unbalance rotation. On the other hand, the torsional vibrations of unbalances can be made use of in practice when magnified by special design measures.

The nonuniformity of rotation of the unbalance can be caused:

(a) by the inconstancy of the gravity force moment of the unbalance about the axis of rotation if the latter is not vertical;

(b) by accelerations of the motion of the rotation axis of the unbalance, except when the centre of gravity of the unbalance describes a circle while moving in it uniformly at the angular velocity of rotation of the unbalance;

(c) by changes in the resistance to the rotation of the unbalance caused by the two factors mentioned or by design and service factors.

Consider the effect of the changes in the gravity force moment of the unbalance about the axis of rotation in the simplest case when the body *1* of the vibration generator (Fig. 74) is motionless and the axis of rotation O of unbalance *2* is horizontal. The centre of gravity A of the unbalance in its stable equilibrium position is situa-

ted on the vertical Ox below the axis of rotation O. We shall determine the position of the unbalance by the angle φ by which the radius-vector $r = OA$ of the unbalance centre of gravity has been displaced from the position of stable equilibrium. The system has one degree of freedom and is described by the differential equation

$$J\ddot{\varphi} + m_0 gr \sin\varphi = M \tag{1}$$

where J = moment of inertia of the unbalanced mass with respect to the axis of rotation
m_0 = unbalanced mass
g = acceleration due to gravity
M = difference between the moment developed by the motor on the unbalance shaft and the moment of the forces opposing the rotation.

Assuming $M = 0$, we obtain the differential equation of the free motion of the physical pendulum:

$$\ddot{\varphi} + \Omega^2 \sin\varphi = 0 \tag{2}$$

where

$$\Omega = \sqrt{\frac{m_0 gr}{J}} \tag{3}$$

in accordance with formula (19), Sec. 6, is the natural frequency of the small oscillations of the unbalance about the position of stable equilibrium. In this case the motion of the points on the unbalance is circulating rather than oscillatory.

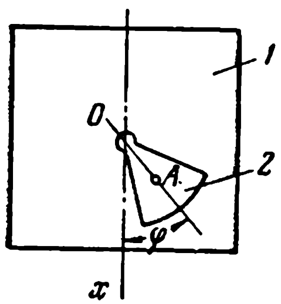


Figure 74

Assuming the initial conditions to be $\varphi = 0$, $\dot{\varphi} = \dot{\varphi}_0$ at $t = 0$ and assuming $\dot{\varphi} > 0$, we can integrate the differential equation (2) as follows:

$$\dot{\varphi} = \sqrt{\dot{\varphi}_0^2 - 2\Omega^2 (1 - \cos\varphi)} \tag{4}$$

It follows from (4) that the angular velocity oscillates between $\dot{\varphi}_{max} = \dot{\varphi}_0$ at $\varphi = 2n\pi$ and $\dot{\varphi}_{min} = \dot{\varphi}_0 \sqrt{1 - k^2}$ at $\varphi = (2n+1)\pi$, $(n = 0, 1, 2, \ldots)$, where

$$k = \frac{2\Omega}{\dot{\varphi}_0} \tag{5}$$

In order to make the unbalance rotate in one direction without stops of finite duration the condition $k < 1$ must be satisfied. In fact the condition $k^2 \ll 1$ holds nearly always.

Equation (4) yields the following function, in the form of a quadrature,

$$t=\int_0^{\varphi}\frac{d\theta}{\sqrt{\dot{\varphi}_0^2-2\Omega^2(1-\cos\theta)}}\tag{6}$$

which is the reciprocal of the second integral of Eq. (2), and which can be reduced by identity transformation to the normal Legendre form of the incomplete elliptic integral of the first kind

$$t=\frac{2}{\dot{\varphi}_0}\int_0^{\frac{\varphi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\tag{7}$$

The second integral φ of differential equation (2) can now be expressed by means of the function known as the *Jacobian amplitude*:

$$\varphi=2\,\mathrm{am}\left(\frac{1}{2}\dot{\varphi}_0 t\right)\tag{8}$$

The Jacobian amplitude is an odd function and can be represented by the sum of a linear function proportional to the function argument and a periodic function which can be expanded in a Fourier series in sine terms

$$\varphi=\omega t+4\sum_{n=1}^{\infty}\frac{\lambda^n}{n(1+\lambda^{2n})}\sin n\omega t\tag{9}$$

where the mean angular velocity of rotation of the unbalance

$$\omega=\frac{\pi\dot{\varphi}_0}{2K(k)}\tag{10}$$

$$\lambda=\exp\left(-\frac{\pi K(\sqrt{1-k^2})}{K(k)}\right)\tag{11}$$

The complete elliptic integrals of the first kind

$$\left.\begin{aligned}K(k)&=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\\ K(\sqrt{1-k^2})&=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-(1-k^2)\sin^2\theta}}\end{aligned}\right\}\tag{12}$$

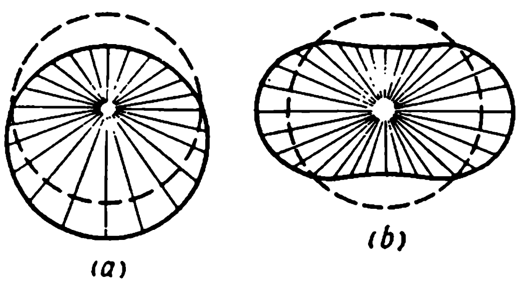
may be taken from reference tables or calculated by making use of expansions in series in even powers of k:

$$\left.\begin{aligned} K(k) &= \frac{\pi}{2}\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+\frac{25}{256}k^6+\dots\right) \\ K\left(\sqrt{1-k^2}\right) &= \ln\frac{4}{k}+\frac{1}{4}\left(\ln\frac{4}{k}-1\right)k^2+ \\ &+\frac{9}{64}\left(\ln\frac{4}{k}-\frac{7}{6}\right)k^4+\frac{25}{256}\left(\ln\frac{4}{k}-\frac{37}{30}\right)k^6+\dots \end{aligned}\right\} \tag{13}$$

Differentiating equality (9) with respect to time, we obtain an expression for the angular velocity of the unbalance in the form of a Fourier series:

$$\dot{\varphi}=\omega\left(1+4\sum_{n=1}^{\infty}\frac{\lambda^n}{1+\lambda^{2n}}\cos n\omega t\right) \tag{14}$$

Thus a Fourier series comprising all harmonics is superposed on the mean angular velocity of rotation of the unbalanced masses. These



(a) (b)

Figure 75

superposed vibrations are usually small. Thus, at $k = 0.1$ (this order of magnitude corresponds to the actual values of k in many centrifugal vibration generators) the amplitude of the first largest harmonic which is only about 0.25% of the mean value of the angular velocity. If $\dot{\varphi}_0$ is reduced by a factor of 5, the ratio of the first harmonic amplitude to the mean angular velocity increases 28 times and amounts to about 7%. If $\dot{\varphi}_0$ is reduced by a factor of 8, the ratio of the amplitude of the first harmonic to the mean angular velocity increases 100 times and amounts to about 25%.

Figure 75*a* shows a circular diagram of the angular velocities of the unbalanced mass plotted according to formula (4). The dotted line circle represents the level of the mean angular velocity and the full line, the running values of the velocity. The scale of the vibrations is strongly exaggerated to visualize them better.

The influence of the acceleration of the rotation axis of the unbalanced mass on its angular velocity will be discussed on the simplest example when the body *1* (Fig. 74) of the generator, because of the ideal restraints imposed on it, has one degree of freedom—translational displacement along the Ox-axis in the absence of gravity, elastic and dissipative connections of the body with the external medium. Choosing as generalized coordinates the displacement x of the body from the mean position and the angle of rotation φ of the unbalanced mass measured from the positive direction of the Ox-axis, we obtain the following expression for the kinetic energy of the system:

$$T=\frac{1}{2}(m_1+m_0)\dot{x}^2+\frac{1}{2}J\dot{\varphi}^2-m_0r\dot{x}\dot{\varphi}\sin\varphi \tag{15}$$

Using relation (15), we now write the differential equations of motion of the system:

$$\left.\begin{aligned}&(m_1+m_0)\ddot{x}-m_0r(\ddot{\varphi}\sin\varphi+\dot{\varphi}^2\cos\varphi)=0\\&J\ddot{\varphi}-m_0r\ddot{x}\sin\varphi=M\end{aligned}\right\} \tag{16}$$

Assuming, as in the preceding case, $M=0$ and eliminating $\ddot{x}$ from Eqs. (16), we obtain

$$\ddot{\varphi}-\alpha^2(\ddot{\varphi}\sin\varphi+\dot{\varphi}^2\cos\varphi)\sin\varphi=0 \tag{17}$$

where

$$\alpha=\sqrt{\frac{(m_0r)^2}{J(m_1+m_0)}} \tag{18}$$

and is always less than 1. Generally, we have $\alpha^2\ll 1$.

We assume the following initial conditions under which the projection of the centre of gravity of the system on the Ox-axis remains motionless: at $t=0$

$$\left.\begin{aligned}&\varphi=0,\ \dot{\varphi}=\dot{\varphi}_0\\&x=-x_0,\ \dot{x}=0\end{aligned}\right\} \tag{19}$$

The first integral of differential equation (17) satisfying these conditions can be presented in the following form:

$$\dot{\varphi}=\frac{\dot{\varphi}_0}{\sqrt{1-\alpha^2\sin^2\varphi}} \tag{20}$$

Expression (20) permits one to obtain a quadrature which is the reciprocal function of the second integral of Eq. (17):

$$t=\frac{1}{\dot{\varphi}_0}\int_0^{\varphi}\sqrt{1-\alpha^2\sin^2\theta}\,d\theta \tag{21}$$

The right-hand quadrature is the normal Legendre's form of the elliptic integral of the second kind $E(\alpha, \varphi)$. Values of φ for any argument $\dot{\varphi}_0 t = E(\alpha, \varphi)$ can be found from reference tables and curves.

Let us expand the right-hand side of equality (20) in a Fourier series

$$\dot{\varphi} = \dot{\varphi}_0 (a_0 - a_2' \cos 2\varphi + a_4' \cos 4\varphi - a_6' \cos 6\varphi + a_8' \cos 8\varphi - \ldots) \tag{22}$$

The coefficients of the series can be presented in the form of power series as follows:

$$\left.\begin{aligned} a_0 &= 1 + \frac{1}{4}\alpha^2 + \frac{9}{64}\alpha^4 + \frac{25}{256}\alpha^6 + \frac{1225}{16,384}\alpha^8 + \ldots \\ a_2' &= \frac{1}{4}\alpha^2 + \frac{3}{16}\alpha^4 + \frac{75}{512}\alpha^6 + \frac{245}{2048}\alpha^8 + \ldots \\ a_4' &= \frac{3}{64}\alpha^4 + \frac{15}{256}\alpha^6 + \frac{245}{4096}\alpha^8 + \ldots \\ a_6' &= \frac{5}{512}\alpha^6 + \frac{35}{2048}\alpha^8 + \ldots \\ a_8' &= \frac{35}{16,384}\alpha^8 + \ldots \end{aligned}\right\} \tag{23}$$

Since the motion is stationary, the constant component in expansion (22) must be equal to the mean angular velocity ω. Hence

$$\dot{\varphi}_0 = \frac{\omega}{a_0} \tag{24}$$

Consequently, series (22) can be rewritten as follows:

$$\dot{\varphi} = \omega - a_2 \cos 2\varphi + a_4 \cos 4\varphi - a_6 \cos 6\varphi + a_8 \cos 8\varphi - \ldots \tag{25}$$

where

$$a_2 = \frac{\omega a_2'}{a_0}; \quad a_4 = \frac{\omega a_4'}{a_0}; \quad a_6 = \frac{\omega a_6'}{a_0}; \quad a_8 = \frac{\omega a_8'}{a_0} \tag{26}$$

Since the amplitudes of the harmonics are small, we may use the approximate equality

$$\varphi = \omega t \tag{27}$$

and rewrite expression (25) as

$$\dot{\varphi} = \omega - a_2 \cos 2\omega t + a_4 \cos 4\omega t - a_6 \cos 6\omega t + a_8 \cos 8\omega t - \ldots \tag{28}$$

Thus the angular velocity of rotation of the unbalanced mass is a periodic function containing even harmonics only. As a rule, α is considerably greater than k for centrifugal vibration generators. Therefore the torsional vibrations of the unbalances caused by the

accelerations of the motion of their axis of rotation are somewhat stronger than those excited by the force of gravity. These vibrations however are usually small: the ratio of the amplitude of the second harmonic, which is the largest, to the mean angular velocity of the rotating unbalanced masses is within the limits 0.2 to 4%. In distinction to the preceding case, this ratio does not vary with the mean rotation velocity since α is independent of it. Figure 75*b* is a diagram of the variation of the angular velocity of the unbalanced mass with account taken of the second harmonic. The designations in the figure are the same as in Fig. 75*a*.

Though the swing of torsional vibrations of the unbalanced mass is usually small, it would be necessary to apply to the shaft an alternating torque of large amplitude to eliminate these vibrations. In fact, if $\dot{\varphi} = \omega = \text{const}$, then $\ddot{\varphi} = 0$ and $\varphi = \omega t$. Inserting these values into the first of Eqs. (16), we obtain the relation which we derived earlier in Chap. 5:

$$\ddot{x} = \frac{m_0 r \omega^2}{m_1 + m_0} \cos \omega t$$

Substituting this relation into the second of Eqs. (16), we obtain

$$M = -\frac{(m_0 r)^2 \omega^2}{2(m_1 + m_0)} \sin 2\omega t$$

This corresponds to the alternating unbalance shaft power

$$N = -\frac{(m_0 r)^2 \omega^3}{2(m_1 + m_0)} \sin 2\omega t$$

This power amplitude in many cases would be several times that of the rated motor power of the vibration machine. The suppression of the small torsional vibrations of the unbalanced mass is therefore far from being a simple matter.

Let us turn now to the effect of the variation of the angular velocity of the unbalanced masses on the motion of the body of the vibration machine. For this purpose we determine from the first of Eqs. (16) the acceleration

$$\ddot{x} = \frac{m_0 r}{m_1 + m_0} (\ddot{\varphi} \sin \varphi + \dot{\varphi}^2 \cos \varphi) \tag{29}$$

Substituting into (29) series (28) for $\dot{\varphi}$ and the time derivative of the series for $\ddot{\varphi}$ and setting, as an approximation, $\sin \varphi = \sin \omega t$, $\cos \varphi = \cos \omega t$, we obtain the following relation:

$$\ddot{x} = \frac{m_0 r \omega^2}{m_1 + m_0} [(1 + p_1 \alpha^2 + p_2 \alpha^4 + \ldots) \cos \omega t + \\ + q_2 \cos 3\omega t + q_3 \cos 5\omega t + \ldots] \tag{30}$$

It follows that, apart from the fundamental tone, the vibrations of the body contain an infinite series of odd harmonics, though their amplitudes are generally small. Thus, the amplitude of the strongest harmonic, the third one, is generally in the range from tenths of one per cent to a few per cent of the fundamental tone amplitude.

34. Multiplication of Vibration Frequency

The superharmonic vibration drive is an interesting example of using in practice the nonuniformity of rotation of unbalances. As can be seen from formula (30), Sec. 33, the vibrations of the working member comprise higher odd harmonics. In fact they also contain even harmonics excited by variations in the gravity force moment of the unbalance about its axis of rotation. One of the harmonics can be very strongly magnified, up to the value required in practice. The third harmonic of the vibrations of the working member is the easiest to magnify.

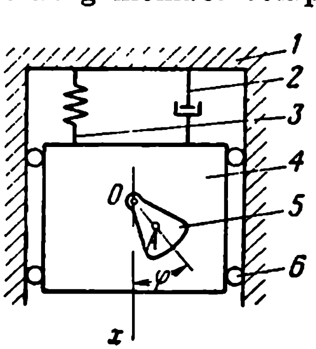


Figure 76

The generation of high-frequency vibrations by centrifugal vibration machines with relatively slow rotation of the unbalanced masses has a number of advantages as compared to the generation of such vibrations by high-speed machines. In particular, the reliability of the vibration machine can be enhanced, the power losses in the bearings of unbalances and the noisiness of the machine are reduced and safety in operation is increased.

The problem of developing a superharmonic vibration drive involves the use of a number of design measures. In order to elucidate the possible approaches to the problem we shall consider a sequence of schematic layouts of increasing complexity. Firstly we turn to the arrangement shown in Fig. 76. The working member *4* is connected to fixed stand *1* by spring *3* and damper *2*. The working member limited by ideal constraints *6* is vibrated by unbalance *5*. The rest of the designations are the same as in Fig. 74.

We choose as generalized coordinates the displacement x of the working member from the position of stable equilibrium and the angle of turning φ of the unbalanced mass and write the differential equation of motion (neglecting the force of gravity, which is insignificant, since we shall be interested in the following discussion in the problem of magnifying the third harmonic component of vibration of the working member):

$$\left.\begin{aligned} (m_1+m_0)\ddot{x}+b\dot{x}+cx-m_0r\,(\ddot{\varphi}\sin\varphi+\dot{\varphi}^2\cos\varphi)=0 \\ J\ddot{\varphi}-m_0r\ddot{x}\sin\varphi=M \end{aligned}\right\} \qquad (1)$$

where b = coefficient of resistance of damper *2*
c = stiffness coefficient of spring *3*
M = constant shaft moment of unbalance.

We now introduce the following dimensionless quantities:

$$\left.\begin{aligned} &\tau=\omega t; \quad \xi=\frac{m_1+m_0}{m_0 r}x; \quad \mu=\frac{M}{J\omega^2} \\ &\gamma=\sqrt{\frac{c}{(m_1+m_0)\omega^2}}; \quad \beta=\frac{b}{2\gamma(m_1+m_0)\omega}; \quad \alpha=\frac{m_0 r}{\sqrt{J(m_1+m_0)}} \end{aligned}\right\} \tag{2}$$

where

$$\omega=\frac{2\pi}{T} \tag{3}$$

is the mean angular velocity of the unbalanced mass (T is the duration of one revolution of the unbalanced mass). Denoting in what follows the derivatives with respect to τ by dots above ξ and φ and substituting the quantities (2) in Eqs. (1), we obtain

$$\left.\begin{aligned} \ddot{\xi}+2\beta\gamma\dot{\xi}+\gamma^2\xi-\ddot{\varphi}\sin\varphi-\dot{\varphi}^2\cos\varphi=0 \\ \ddot{\varphi}-\alpha^2\ddot{\xi}\sin\varphi=\mu \end{aligned}\right\} \tag{4}$$

Let us express the angle φ in the form of the sum of the linear and oscillating parts:

$$\varphi=\tau+\psi \tag{5}$$

whence

$$\dot{\varphi}=1+\dot{\psi}, \quad \ddot{\varphi}=\ddot{\psi} \tag{6}$$

Inserting (5) and (6) into Eqs. (4), we obtain

$$\left.\begin{aligned} \ddot{\xi}+2\beta\gamma\dot{\xi}+\gamma^2\xi=\ddot{\psi}\sin(\tau+\psi)+(1+\dot{\psi})^2\cos(\tau+\psi) \\ \ddot{\psi}-\alpha^2\ddot{\xi}\sin(\tau+\psi)=\mu \end{aligned}\right\} \tag{7}$$

In the following treatment we assume that the angle ψ, its derivatives $\dot{\psi}$, $\ddot{\psi}$ and the parameter α^2 are small compared to unity. We rewrite Eq. (7), retaining only small terms up to the first order inclusive:

$$\left.\begin{aligned} \ddot{\xi}+2\beta\gamma\dot{\xi}+\gamma^2\xi=\cos\tau+(\ddot{\psi}-\psi)\sin\tau+2\dot{\psi}\cos\tau \\ \ddot{\psi}=\alpha^2\ddot{\xi}\sin\tau+\mu \end{aligned}\right\} \tag{8}$$

We shall further make use of the method of successive approximations. We assume as the first approximation

$$\psi^*=0 \tag{9}$$

and substituting this approximation into the first of equations (8), we obtain the first approximation to ξ:

$$\xi^* = \xi^*_{1a}\cos(\tau-\tau^*_1) \tag{10}$$

where

$$\left.\begin{aligned} \xi^*_{1a} &= \frac{1}{\sqrt{(\gamma^2-1)^2+4\beta^2\gamma^2}} \\ \tau^*_1 &= \tan^{-1}\frac{2\beta\gamma}{\gamma^2-1} \end{aligned}\right\} \tag{11}$$

The figures in the subscripts of the amplitude and initial phase denote here and further the order number of the harmonic. The superscript (*) denotes the first approximation; the second approximation is written without it.

To obtain the next approximation to ψ we insert (10) into the second of equations (8):

$$\ddot{\psi} = -\frac{1}{2}\alpha^2\xi^*_{1a}\sin(2\tau-\tau^*_1) - \frac{1}{2}\alpha^2\xi^*_{1a}\sin\tau^*_1 + \mu \tag{12}$$

The condition for the periodicity of solution (12) is the equality

$$\mu^* = \frac{1}{2}\alpha^2\xi^*_{1a}\sin\tau^*_1 \tag{13}$$

whose dimensionless form represents the power balance since the quantity μ is proportional to the mean power required to sustain the vibrations.

Making use of this condition, we rewrite Eq. (12) in the following form:

$$\ddot{\psi} = -\frac{1}{2}\alpha^2\xi^*_{1a}\sin(2\tau-\tau^*_1) \tag{14}$$

whence

$$\left.\begin{aligned} \dot{\psi} &= \frac{1}{4}\alpha^2\xi^*_{1a}\cos(2\tau-\tau^*_1) \\ \psi &= \frac{1}{8}\alpha^2\xi^*_{1a}\sin(2\tau-\tau^*_1) \end{aligned}\right. \tag{15}$$

To obtain the second approximation to ξ we substitute relations (14) and (15) into the first of Eqs. (8), neglecting small quantities of higher order than the first in α^2:

$$\ddot{\xi} + 2\beta\gamma\dot{\xi} + \gamma^2\xi = \left(1 - \frac{\alpha^2\xi^*_{1a}}{16}\cos\tau^*_1\right)\cos(\tau-\chi) + \frac{9}{16}\alpha^2\xi^*_{1a}\cos(3\tau-\tau^*_1) \tag{16}$$

where

$$\chi = \tan^{-1}\left(\frac{\alpha^2\xi^*_{1a}}{16}\sin\tau^*_1\right) \tag{17}$$

The second approximation to ξ is furnished by the integral of differential equation (16):

$$\xi = \xi_{1a} \cos(\tau - \tau_1) + \xi_{3a} \cos(3\tau - \tau_3) \tag{18}$$

where

$$\xi_{1a} = \xi_{1a}^* \left(1 - \frac{\alpha^2}{16} \cos \tau_1^*\right); \quad \tau_1 = \tau_1^* + \chi \tag{19}$$

$$\xi_{3a} = \frac{9\alpha^2 \xi_{1a}^*}{16 \sqrt{(\gamma^2 - 9)^2 + 36\beta^2\gamma^2}}; \quad \tau_3 = \tau_1^* + \tan^{-1} \frac{6\beta\gamma}{\gamma^2 - 9} \tag{20}$$

Substituting the integral (18) into the second of equations (8), we obtain the second approximation to the power balance

$$\mu = \frac{1}{2} \alpha^2 \xi_{1a} \sin \tau_1 \tag{21}$$

Let us discuss the results. Expressions (18), (19) and (21) show that the amplitude of the fundamental (first) tone, its initial phase and the power balance differ but slightly from the case when $\dot{\varphi} = \text{const}$. The response characteristics of the first harmonic are close to those discussed in Section 13 which are the result of excitation by a centrifugal force of constant modulus[1].

A feature of the phase response relation $\tau_1(\gamma)$ is that with $\gamma = 1$ the initial phase $\tau_1 > \pi/2$.

The amplitude of the third harmonic ξ_{3a} is proportional to α^2 and the first-harmonic amplitude ξ_{1a}^*. The third harmonic has two resonances: together with the first harmonic in the neighbourhood of $\gamma = 1$ and also in the neighbourhood of $\gamma = 3$. The initial phase of the third harmonic τ_3 varies within the segment $0 \leqslant \tau_3 \leqslant 2\pi$, the changes inside the zone of each resonance being rapid; at $\gamma = 1$ the initial phase $\tau_3 < 3\pi/2$ and at $\gamma = 3$ the initial phase $\tau_3 > \pi/2$.

Since in the neighbourhood of $\gamma = 1$ the first and the third harmonics have simultaneous resonances, the ratio $\xi_{3a} : \xi_{1a}$ remains nearly constant. But in the neighbourhood of $\gamma = 3$ only the third harmonic is resonant and this results in an increase in the ratio $\xi_{3a} : \xi_{1a}$. Thus the increase in the ratio of the amplitude of the third harmonic of vibration of the working member to the first-harmonic amplitude can be attained by increasing the parameter α or by the third harmonic approaching resonance in the neighbourhood of $\gamma = 3$ as well as by increasing the magnification factor of this resonance, i.e., by reducing the damping ratio β.

Figure 77*a-d* shows the response curves for the first and third harmonics. The arrangement illustrated in Fig. 76 does not ensure the isolation of the stand from vibrations with a considerable range

[1] The frequency ratio γ is identical with the ratio γ_* in Section 13.

in the neighbourhood of $\gamma = 3$, i.e., with a spring of large stiffness which, according to the fourth of expressions (2), is

$$c = \gamma^2 (m_1 + m_0) \omega^2$$

Another essential shortcoming of this arrangement is that the third harmonic resonance in the neighbourhood of $\lambda = 3$ is much less pronounced than in the neighbourhood of $\gamma = 1$ since the amplitude ξ_{3a} is proportional to the amplitude ξ_{1a} and the latter becomes very small in the neighbourhood of $\gamma = 3$ because of the high stiffness of the spring.

The arrangement illustrated in Fig. 78 proves to be more advantageous. Here the body *4* of the vibration generator is connected

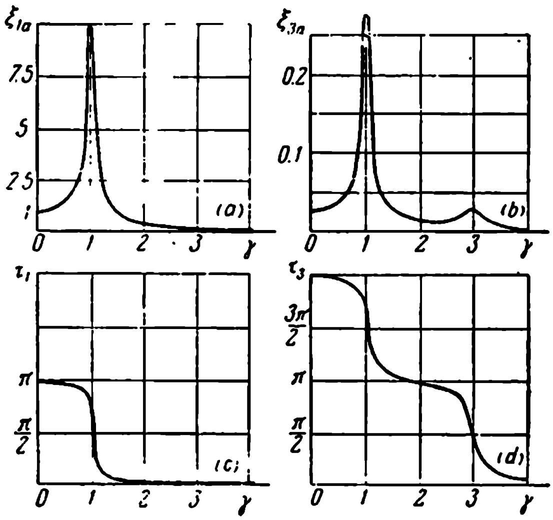


Figure 77

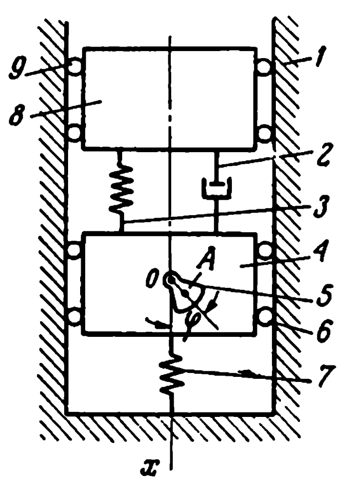


Figure 78

to the fixed foundation *1* by a very compliant spring *7*. Spring *3* and damper *2* connect the body with a second body *8* which we shall agree to call the working member[1]. The body and the working member can perform only translational motion along the x-axis because of ideal constraints *6* and *9*. Unbalance *5* rotates about the axis O which is rigidly connected to the body.

We select as generalized coordinates the displacements $x^{(1)}$ and $x^{(2)}$ from the equilibrium positions of the body and working member, respectively, and the angle of rotation φ of the unbalanced mass

[1] In fact, the body *4* itself may serve as a working member and the second body as a reactive element. Cases are possible when both bodies, *4* and *8*, are working members.

measured from the positive direction of the x-axis. The kinetic energy of the system

$$T=\frac{1}{2}m_2(\dot{x}^{(2)})^2+\frac{1}{2}(m_1+m_0)(\dot{x}^{(1)})^2+\frac{1}{2}J\dot{\varphi}^2-m_0r\dot{x}^{(1)}\dot{\varphi}\sin\varphi \qquad (22)$$

The potential energy (neglecting the force of gravity and the stiffness of the supporting spring *7*)

$$\Pi=\frac{1}{2}c\,(x^{(1)}-x^{(2)})^2 \qquad (23)$$

and the dissipative function

$$\Phi=\frac{1}{2}b\,(\dot{x}^{(1)}-\dot{x}^{(2)})^2 \qquad (24)$$

where m_1 = mass of the generator body
m_2 = mass of the working member
c = coefficient of stiffness of spring *3*
b = resistance coefficient of damper *2*; the rest of the notations are the same as in relation (15), Sec. 33.

On the basis of equalities (22) through (24) we set up the differential equations of motion:

$$\left.\begin{aligned}
&m_2\ddot{x}^{(2)}-b(\dot{x}^{(1)}-\dot{x}^{(2)})-c(x^{(1)}-x^{(2)})=0\\
&(m_1+m_2)\ddot{x}^{(1)}+b(\dot{x}^{(1)}-\dot{x}^{(2)})+c(x^{(1)}-x^{(2)})-\\
&\quad-m_0r(\ddot{\varphi}\sin\varphi+\dot{\varphi}^2\cos\varphi)=0\\
&J\ddot{\varphi}-m_0r\ddot{x}^{(1)}\sin\varphi=M
\end{aligned}\right\} \qquad (25)$$

We introduce the following dimensionless quantities:

$$\left.\begin{aligned}
&\lambda=\frac{m_1+m_0}{m_2+m_1+m_0};\quad \beta=\frac{b}{2\gamma\lambda m_2\omega};\quad \gamma=\sqrt{\frac{c}{\lambda m_2\omega^2}}\\
&\xi^{(1)}=\frac{m_1+m_0}{m_0r}x^{(1)};\quad \xi^{(2)}=\frac{m_1+m_0}{m_0r}x^{(2)}
\end{aligned}\right\} \qquad (26)$$

the quantities τ, μ, α, ψ defined by equalities (2), (3), and (5), and the variable

$$\xi=\xi^{(1)}-\xi^{(2)} \qquad (27)$$

and rewrite Eq. (25) in the following form (the dots are used to denote differentiation with respect to τ), retaining only the terms within the first order of smallness:

$$\left.\begin{aligned}
&\ddot{\psi}=\alpha^2\ddot{\xi}^{(1)}\sin\tau+\mu\\
&\ddot{\xi}+2\beta\gamma\dot{\xi}+\gamma^2\xi=\cos\tau+(\ddot{\psi}-\psi)\sin\tau+2\dot{\psi}\cos\tau\\
&\ddot{\xi}^{(1)}=\ddot{\xi}+2\beta\gamma\lambda\dot{\xi}+\gamma^2\lambda\xi
\end{aligned}\right\} \qquad (28)$$

Taking expression (9) as the first approximation to ψ, we obtain the first approximation to ξ determined by (10) and (11). We proceed to determine the first approximation to $\xi^{(1)}$ from the third of Eqs. (28):

$$\xi^{(1)*} = \xi_{1a}^{(1)*} \cos(\tau - \tau_1^{(1)*}) \tag{29}$$

where

$$\left.\begin{aligned} \xi_{1a}^{(1)*} &= \xi_{1a}^{*} \sqrt{(\gamma^2\lambda - 1)^2 + 4\beta^2\gamma^2\lambda^2} \\ \tau_1^{(1)*} &= \tau_1^{*} + \pi - \tan^{-1}\frac{2\beta\gamma\lambda}{\gamma^2\lambda - 1} \end{aligned}\right\} \tag{30}$$

The second approximation to ψ is determined from the first of Eqs. (28) and is expressed by

$$\psi = \frac{1}{8}\alpha^2\xi_{1a}^{(1)*} \sin(2\tau - \tau_1^{(1)*}) \tag{31}$$

with the condition of periodicity

$$\mu^* = \frac{1}{2}\alpha^2\xi_{1a}^{(1)*} \sin\tau_1^{(1)*} \tag{32}$$

The second approximation to ξ is determined by expression (18) where

$$\left.\begin{aligned} \xi_{1a} &= \xi_{1a}^{*}\left(1 - \frac{\alpha^2}{16}\xi_{1a}^{(1)*}\right)\cos\tau_1^{(1)*} \\ \tau_1 &= \tau_1^{*} + \tan^{-1}\left(\frac{\alpha^2}{16}\xi_{1a}^{(1)*}\sin\tau_1^{(1)*}\right) \\ \xi_{3a} &= \frac{9\alpha^2\xi_{1a}^{(1)*}}{16\sqrt{(\gamma^2 - 9)^2 + 36\beta^2\gamma^2}} \\ \tau_3 &= \tau_1^{*} + \tan^{-1}\frac{6\beta\gamma}{\gamma^2 - 9} \end{aligned}\right\} \tag{33}$$

The second approximation to $\xi^{(1)}$ is expressed by the equality

$$\xi^{(1)} = \xi_{1a}^{(1)}\cos(\tau - \tau_1^{(1)}) + \xi_{3a}^{(1)}\cos(3\tau - \tau_3^{(1)}) \tag{34}$$

where

$$\left.\begin{aligned} &\xi_{1a}^{(1)} = \xi_{1a}\sqrt{(\gamma^2\lambda - 1)^2 + 4\beta^2\gamma^2\lambda^2}; \quad \tau_1^{(1)} = \tau_1 + \pi - \tan^{-1}\frac{2\beta\gamma\lambda}{\gamma^2\lambda - 1} \\ &\xi_{3a}^{(1)} = \xi_{3a}\sqrt{(\gamma^2\lambda - 9)^2 + 36\beta^2\gamma^2\lambda^2}; \quad \tau_3^{(1)} = \tau_3 + \pi - \tan^{-1}\frac{6\beta\gamma\lambda}{\gamma^2\lambda - 9} \end{aligned}\right\} \tag{35}$$

The second approximation to $\xi^{(2)}$ is represented, in accordance with expression (27), by the equality

$$\xi^{(2)} = \xi_{1a}^{(2)}\cos(\tau - \tau_1^{(2)}) + \xi_{3a}^{(2)}\cos(3\tau - \tau_3^{(2)}) \tag{36}$$

where

$$\left.\begin{aligned} \xi_{na}^{(2)} &= \sqrt{(\xi_{na}^{(1)}\cos\tau_n^{(1)} - \xi_{na}\cos\tau_n)^2 + (\xi_{na}^{(1)}\sin\tau_n^{(1)} - \xi_{na}\sin\tau_n)^2} \\ \tau_n &= \tan^{-1}\frac{\xi_{na}^{(1)}\sin\tau_n^{(1)} - \xi_{na}\sin\tau_n}{\xi_{na}^{(1)}\cos\tau_n^{(1)} - \xi_{na}\cos\tau_n}, \quad (n = 1,\ 3) \end{aligned}\right\} \tag{37}$$

A serious shortcoming of the arrangement discussed above and also of the preceding one is that the possibilities of increasing the coefficient α are limited, particularly when it is required to vibrate large masses. True, the arrangement shown in Fig. 78 permits one to obtain a greater ratio of the amplitude of the third harmonic of the vibration displacement of the working member to that of the first harmonic and a higher magnification of the third-harmonic amplitude in the neighbourhood of $\gamma = 3$.

The arrangement shown in Fig. 79 offers much wider possibilities. The figure shows a centred system with a pendulum vibration generator whose parameters are chosen, in accordance with the recommendations stated in Section 31, in such a way that no reactions appear on pivot O of pendulum *2* in a direction at right angles to the central x-axis. The pendulum is hinged to frame *6* which is supported by fixed baseplate *4* by means of springs *5* of large compliance. The axis A of the unbalance is rigidly connected with the pendulum. Springs *7* and damper *1* connect the frame with working member *8*. Since the system is centred, the centres of gravity of the working member, frame, pendulum and unbalance are all on the central x-axis, when they are in the equilibrium position. The resultant of the reactions of springs *7* and the reaction of damper *1* act constantly along this x-axis.

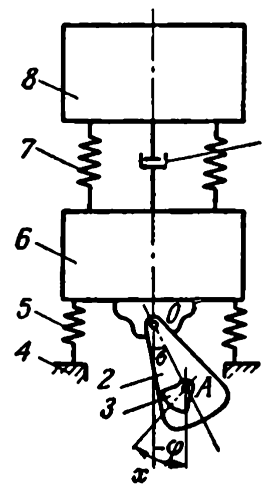


Figure 79

The pendulum performs small oscillations under the action of rotation of the unbalanced mass and the frame and working member vibrate in a translational motion in the x-direction. We select as generalized coordinates the displacements $x^{(2)}$ and $x^{(1)}$ of the working member and the frame, respectively, relative to the mean position under steady-state vibrations, the displacement angle σ of the pendulum and the angle of rotation φ of the unbalanced mass both measured from the equilibrium position.

We introduce the following notations: $m_1' =$ pendulum mass, $m_1'' =$ frame mass; $J_1 =$ pendulum moment of inertia with respect to the axis of swinging, the moment of inertia of the unbalance being taken into account by placing an equal point mass at the axis of rotation; $a =$ distance from the axis of swinging of pendulum

to its centre of gravity; l = distance from the axis of swinging of pendulum to the axis of rotation of unbalanced mass. The rest of the notations are the same as those for the arrangement shown in Fig. 78. Neglecting the force of gravity and the reactions of springs 5, we can now write the following differential equations of motion of the system:

$$\left.\begin{aligned}
&(m_1''+m_1'+m_0)\frac{d^2x^{(1)}}{dt^2}+b\left(\frac{dx^{(1)}}{dt}-\frac{dx^{(2)}}{dt}\right)+c\left(x^{(1)}-x^{(2)}\right)-\\
&\quad-(m_1'a+m_0l)\left[\frac{d^2\sigma}{dt^2}\sin\sigma+\left(\frac{d\sigma}{dt}\right)^2\cos\sigma\right]-\\
&\quad-m_0r\left[\frac{d^2\varphi}{dt^2}\sin\varphi+\left(\frac{d\varphi}{dt}\right)^2\cos\varphi\right]=0\\
&m_2\frac{d^2x^{(2)}}{dt^2}-b\left(\frac{dx^{(1)}}{dt}-\frac{dx^{(2)}}{dt}\right)-c\left(x^{(1)}-x^{(2)}\right)=0\\
&J_1\frac{d^2\sigma}{dt^2}-(m_1'a+m_0l)\frac{d^2x^{(1)}}{dt^2}\sin\sigma+\\
&\quad+m_0rl\left[\frac{d^2\varphi}{dt^2}\cos(\varphi-\sigma)-\left(\frac{d\varphi}{dt}\right)^2\sin(\varphi-\sigma)\right]=0\\
&J\frac{d^2\varphi}{dt^2}-m_0r\frac{d^2x^{(1)}}{dt^2}\sin\varphi+\\
&\quad+m_0rl\left[\frac{d^2\sigma}{dt^2}\cos(\varphi-\sigma)+\left(\frac{d\sigma}{dt}\right)^2\sin(\varphi-\sigma)\right]=M
\end{aligned}\right\}\qquad(38)$$

We introduce the following dimensionless parameters:

$$\left.\begin{aligned}
&\lambda=\frac{m_1''+m_1'+m_0}{m_2+m_1''+m_1'+m_0};\qquad \alpha=\frac{m_0r}{\sqrt{J(m_1''+m_1'+m_0)}}\\
&\xi^{(1)}=\frac{m_1''+m_1'+m_0}{m_0r}x^{(1)};\qquad \xi^{(2)}=\frac{m_1''+m_1'+m_0}{m_0r}x^{(2)}\\
&\varepsilon=\frac{m_0r}{m_1'a+m_0l};\qquad i=\frac{J}{J_1};\qquad \rho=\sqrt{\frac{m_0rl}{J}}
\end{aligned}\right\}\qquad(39)$$

and also τ, μ, ψ defined by relations (2), (3), and (5) and β, γ, ξ given by equalities (26) and (27).

To simplify matters, we assume that ψ, $\dot\psi$, $\ddot\psi$, σ, $\dot\sigma$, $\ddot\sigma$, α^2, i and ε are small quantities of the same order and retain small terms up to the first order inclusive. As a result, we obtain

$$\left.\begin{aligned}
&\ddot\psi=-\rho^2\ddot\sigma\cos\tau+\alpha^2\ddot\xi^{(1)}\sin\tau+\mu\\
&\ddot\sigma=\rho^2 i\sin\tau\\
&\ddot\xi+2\beta\gamma\dot\xi+\gamma^2\xi=\cos\tau+(\ddot\psi-\psi)\sin\tau+2\dot\psi\cos\tau+\frac{1}{\varepsilon}(\ddot\sigma\sigma+\dot\sigma^2)\\
&\ddot\xi^{(1)}=\ddot\xi+2\beta\gamma\lambda\dot\xi+\gamma^2\lambda\xi
\end{aligned}\right\}\qquad(40)$$

We take again expression (9) as the first approximation to ψ. From the second of Eqs. (40) we obtain:

$$\sigma = -\rho^2 i \sin\tau \tag{41}$$

The third of equations (40) now takes the form

$$\ddot{\xi} + 2\beta\gamma\dot{\xi} + \gamma^2\xi = \cos\tau - \frac{\rho^4 i^2}{\varepsilon}\cos 2\tau \tag{42}$$

Hence the first approximation to ξ is

$$\xi^* = \xi^*_{1a}\cos(\tau - \tau^*_1) + \xi^*_{2a}\cos(2\tau - \tau^*_2) \tag{43}$$

where ξ^*_{1a} and τ^*_1 are defined by expressions (11),

$$\xi^*_{2a} = \frac{\rho^4 i^2}{\varepsilon\sqrt{(\gamma^2-4)^2+16\beta^2\gamma^2}}; \quad \tau^*_2 = \tan^{-1}\frac{4\beta\gamma}{\gamma^2-4} \tag{44}$$

The fourth of equations (40) yields the first approximation:

$$\xi^{(1)*} = \xi^{(1)*}_{1a}\cos(\tau - \tau^{(1)*}_1) + \xi^{(1)*}_{2a}\cos(2\tau - \tau^{(1)*}_2) \tag{45}$$

where $\xi^{(1)*}_{1a}$ and $\tau^{(1)*}_1$ are calculated from formulas (30),

$$\left.\begin{aligned} \xi^{(1)*}_{2a} &= \xi^*_{2a}\sqrt{(\gamma^2\lambda-4)^2+16\beta^2\gamma^2\lambda^2} \\ \tau^{(1)*}_2 &= \tau^*_2 + \pi - \tan^{-1}\frac{4\beta\gamma\lambda}{\gamma^2\lambda-4} \end{aligned}\right\} \tag{46}$$

The second approximation to ψ is found from the first of equations (40) upon inserting into it the values given by (41) and (45):

$$\psi = \psi_{2a}\sin(2\tau - \theta_2) \tag{47}$$

where

$$\begin{aligned} \psi_{2a} &= \frac{1}{8}\sqrt{\rho^8 i^2 + \alpha^4(\xi^{(1)*}_{1a})^2} \\ \theta_2 &= \tan^{-1}\frac{\alpha^2\xi^{(1)*}_{1a}\sin\tau^{(1)*}_1}{\alpha^2\xi^{(1)*}_{1a}\cos\tau^{(1)*}_1 - \rho^4 i} \end{aligned} \tag{48}$$

The power balance can be calculated from formula (32). We now find the second approximation to ξ from the third of equations (40), making use of relations (41) and (47):

$$\xi = \xi_{1a}\cos(\tau - \tau_1) + \xi_{2a}\cos(2\tau - \tau_2) + \xi_{3a}\cos(3\tau - \tau_3) \tag{49}$$

where ξ_{2a} and τ_2 can be determined by using equalities (44) and ξ_{1a}, τ_1, ξ_{3a}, τ_3 by using the following equalities

$$\xi_{1a} = \xi^*_{1a}\left(1 - \frac{1}{2}\psi_{2a}\cos\theta_2\right); \quad \tau_1 = \tau^*_1 + \tan^{-1}\left(\frac{1}{2}\psi_{2a}\sin\theta_2\right)$$

$$\xi_{3a} = \frac{9\psi_{2a}}{2\sqrt{(\gamma^2-9)^2+36\beta^2\gamma^2}}; \quad \tau_3 = \tau^*_1 + \tan^{-1}\frac{6\beta\gamma}{\gamma^2-9}$$

Further, from the fourth of equations (40) we obtain

$$\xi^{(1)} = \xi_{1a}^{(1)} \cos (\tau - \tau_1^{(1)}) + \xi_{2a}^{(1)} \cos (2\tau - \tau_2^{(1)}) + \xi_{3a}^{(1)} \cos (3\tau - \tau_3^{(1)}) \quad (50)$$

where

$$\left.\begin{aligned} \xi_{na}^{(1)} &= \xi_{na} \sqrt{(\gamma^2\lambda - n^2)^2 + 4n^2\beta^2\gamma^2\lambda^2} \\ \tau_n^{(1)} &= \tau_n + \pi - \tan^{-1} \frac{2n\beta\gamma\lambda}{\gamma^2\lambda - n^2}, \quad (n = 1, 2, 3) \end{aligned}\right\} \quad (51)$$

Finally, in accordance with equality (27)

$$\xi^{(2)} = \xi_{1a}^{(2)} \cos (\tau - \tau_1^{(2)}) + \xi_{2a}^{(2)} \cos (2\tau - \tau_2^{(2)}) + \xi_{3a}^{(2)} \cos (3\tau - \tau_3^{(2)}) \quad (52)$$

where $\xi_{na}^{(2)}$ and $\tau_n^{(2)}$ ($n = 1, 2, 3$) are determined from formulas (37).

Thus, the working member performs vibrations comprising, within the second approximation, the first three harmonics. The rest of the harmonics are small quantities of higher orders outside their resonance zones. The relations obtained indicate that within their resonance zones the amplitudes of the second and third harmonics can become large quantities.

35. Limitations Imposed by the Motor. Steady-State and Transient Operating Conditions

In the preceding sections the motions of the vibrating systems were treated without taking into account the limited capacity of the motor as the source of the energy required to sustain the vibrations. Where the moment developed by the motor on the shaft of the unbalance was introduced into the differential equations of motion (for instance, in discussing the problem of vibration frequency multiplication, Section 34) it was presumed that the magnitude of the moment corresponds exactly to the conditions for periodicity obtained for the given mean angular velocity of the unbalanced mass. Actually the mean angular velocity cannot be specified arbitrarily. It takes a strictly definite value depending on the properties of the vibrating system, the capacity of the motor and the conditions of bringing the system to operating conditions.

The problems of the interaction between the vibrating system and the motor have been studied by many investigators. The most systematic and complete treatment of the limitations imposed by the motor is due to V. Kononenko. In respect of steady-state motion conditions discussed in this section the problems of the interaction between vibrating system and motor acquire much practical importance in designing resonance vibration machines.

We shall consider the simplest arrangement shown in Fig. 76, assuming the shaft of the driving motor to be rigidly connected with the unbalanced-mass shaft. The motion of this system is descri-

bed by differential equations (1), Sec. 34. The moment on the right-hand side of the second of equations (1) developed by the motor on the unbalanced-mass shaft[1] was considered constant. The moment actually depends on the conditions of motion of the system.

The static characteristics of the motors are known; they represent the relations between the motor torque (or the power developed) and the angular velocity. These characteristics are determined for constant angular velocities and that is why they are called *static*. Thus, the static characteristic of the motor may be represented by either of the relations

$$M = M(\omega), \quad N = N(\omega) = \omega M(\omega) \tag{1}$$

where M = motor torque at the unbalanced-mass shaft minus the moment of the forces opposing the rotation of the unbalanced mass

N = power developed by the motor minus the power required to overcome the resistance to the rotation of the unbalanced mass

ω = (constant) angular velocity of rotation of the unbalanced mass.

On the other hand, one can construct the static characteristic of the resistance offered by the vibrating system which is the relation between the angular velocity of the unbalanced mass and the active power (cf. Section 23), i.e., the mean value of the power N_{mean} required to sustain the vibrations of the system or the relation between the mean value of the moment M_{mean} corresponding to N_{mean} and the angular velocity of the shaft:

$$M_{mean} = M_{mean}(\omega), \quad N_{mean} = N_{mean}(\omega) = \omega M_{mean}(\omega) \tag{2}$$

Both these relations must be obtained for constant angular velocities.

To ensure steady-state vibration conditions at constant angular velocity of the unbalanced masses it is necessary that the following power balance be fulfilled:

$$\omega M(\omega) = \omega M_{mean}(\omega) \tag{3}$$

The solution of Eq. (3) furnishes the angular velocity at which steady-state vibration conditions can be attained. It should however be noted that condition (3) is not sufficient. First of all, Eq. (3) may have more than one real and positive root and it will not be clear which of the angular velocities obtained by solving the equation will be realized. On the other hand, not each position of dynamic equilibrium of the system described by the power balance is stable.

The full curve in Fig. 80*a* represents the static characteristic

[1] To be more exact, it is the difference between the moment developed by the motor and that opposing the rotation of the unbalanced mass.

of the motor $M(\omega)$, and the dotted curve is the static characteristic of the resistance of the vibrating system $M_{mean}(\omega)$. The curves intersect at points *1*, *2*, *3* where condition (3) is satisfied. The abscissas of these points ω_1, ω_2, ω_3 are roots of Eq. (3).

Suppose that the system is in a state corresponding to point *1*. Let us add to the angular velocity ω_1 a small positive increment $\Delta\omega$. For this state the motor moment $M(\omega_1 + \Delta\omega)$ is equal to the ordinate of point $1'_+$, i.e., to the segment $a_+1'_+$, and the resisting moment $M_{mean}(\omega_1 + \Delta\omega)$ is equal to the ordinate of point $1''_+$,

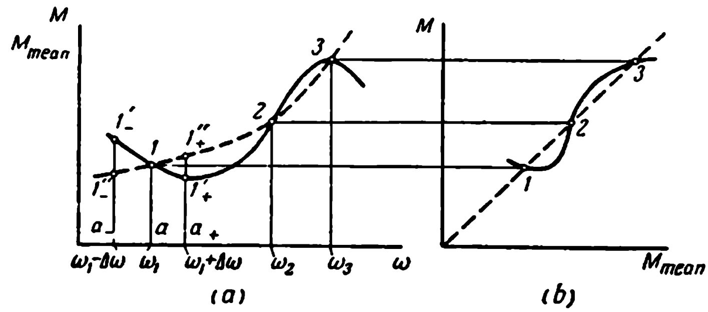


Figure 80

i.e., to the segment $a_+1''_+$. Since $M(\omega_1 + \Delta\omega) < M_{mean}(\omega_1 + \Delta\omega)$ the rotation will slow down and the system will return to point *1*. If a small negative increment $-\Delta\omega$ is imparted to the angular velocity ω_1, then, as shown in Fig. 80*a*, we shall have $M(\omega_1 - \Delta\omega) > M_{mean}(\omega_1 - \Delta\omega)$ and the rotation will be accelerated and the system will again revert to point *1*. Hence the dynamic equilibrium represented by point *1* is stable.

In a similar way we find that the dynamic equilibrium at point *3* is also stable. Point *2* is unstable since with increasing angular velocity the difference $M(\omega_1 + \Delta\omega) - M_{mean}(\omega_1 + \Delta\omega) > 0$ and this causes the angular velocity to rise further and shift from point *2*. If the angular velocity decreases, we have $M(\omega_1 - \Delta\omega) - M_{mean}(\omega_1 - \Delta\omega) < 0$, which leads to a further fall of the velocity and the displacement from point *2*.

It is readily seen that the condition of the stability of the dynamic equilibrium of the system can be expressed by the inequality

$$\left(\frac{dM}{d\omega}\right)_{\omega=\omega_i} < \left(\frac{dM_{mean}}{d\omega}\right)_{\omega=\omega_i} \tag{4}$$

where ω_i is the root of Eq. (3)[1].

Which of the stable points will be realized depends on the conditions under which the system is brought to the operating conditions.

[1] It is presumed that the static characteristics of the motor and the vibrating system intersect without touching.

Thus, if the angular velocity $\omega > \omega_2$ is somehow imparted to a system corresponding to Fig. 80*a*, the system will move on to equilibrium point *3*; but if the angular velocity $\omega < \omega_2$ is imparted to the system, it will go to equilibrium point *1*.

Taking ω as a parameter, we can give inequality (4) the form

$$\left(\frac{dM}{dM_{mean}}\right)_{M=M_{mean}} < 1 \tag{5}$$

Figure 80*b* illustrates the curve $M = M\,(M_{mean})$. The dotted line bisects the angle between the coordinate axes and is represented by the expression

$$\frac{dM}{dM_{mean}} = 1$$

Points *1* and *3* are stable since they satisfy condition (5); on the contrary, point *2* is unstable.

In discussing the changes in the state of the system we assumed the change of the angular velocity to be sufficiently slow to allow the use of static characteristics.

Let us return to the system illustrated in Fig. 76. With the condition $\dot{\varphi} = \omega = \text{const}$ and taking into account the first of equalities (1) the differential equations (1), Sec. 34, of the motion of the system take the form

$$\left.\begin{aligned} (m_1 + m_0)\,\ddot{x} + b\dot{x} + cx &= m_0 r\omega^2 \cos \omega t \\ -\,m_0 r\ddot{x} \sin \omega t &= M\,(\omega) + M'\,(t) \end{aligned}\right\} \tag{6}$$

where $M'\,(t)$ is the alternating reactive moment required to keep the angular velocity constant (cf. Section 33).

We integrate the first of equations (6) and substitute the solution obtained into the second equation. Equating now the constant components on the left- and right-hand sides, we can write the equation of power balance corresponding to relation (3):

$$(m_0 r)^2\, b\omega^5 - 2\,[(c - m\omega^2)^2 + b^2\omega^2]\, M\,(\omega) = 0 \tag{7}$$

On solving this equation by one of the approximate methods we obtain its real and positive roots ω_i. The static characteristic of the resistance of the vibrating system in this case is

$$M_{mean} = \frac{(m_0 r)^2\, b\omega^5}{2\,[(c - m\omega^2)^2 + b^2\omega^2]} \tag{8}$$

Making use of condition (4), we determine the stable points.

The motors are usually started at $\omega = 0$. In a system corresponding to Fig. 80 this will lead to the steady operation represented by point *1*. The motor is incapable (at least with slowly increasing angular velocity) of bringing the system to point *3*. In order to attain this point it is necessary to impart by some means to the unbalanced

mass the angular velocity $\omega_1 > \omega_2$. Quite often the motors have adjustment devices for changing their static characteristics. Such devices are preferably made use of at starting.

Figure 81 shows a family of motor static characteristics (full curves). The curve *1* corresponds to the working conditions, and the curves *2* and *3* can be realized for a short time. The dotted curve represents the static characteristic of the resistance of the system. In order to attain the working point A one must change over to the static characteristic *3* of the motor, raise the speed of the system up to point A' and then switch the motor over to the working conditions and, by reducing the number of revolutions, come to point A.

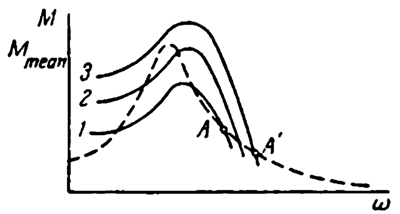


Figure 81

Usually the working point lies on the descending portion of the static characteristic. In the neighbourhood of the normal operation point a linear representation of the static characteristic is often used:

$$M = M_0 - k_i\omega \tag{9}$$

where

$$M_0 = M_i + k_i\omega_i; \quad k_i = -\left(\frac{dM}{d\omega}\right)_{\omega=\omega_i} \tag{10}$$

M_i and ω_i being the values of the motor moment and angular velocity, respectively, at the normal operation (working) point.

The interaction between the motor and the vibrating system leads to the appearance of nonlinear effects in systems that would be linear with a motor of unlimited power and absolutely rigid characteristic, i.e., with $dM/d\omega = -\infty$ and $\omega = \text{const}$. One of the nonlinear effects is Sommerfeld's effect the essence of which consists in that the unbalanced-mass angular velocity changes by jumps with smooth adjustment of the motor voltage. If one adjusts the motor to higher speed, then the working point moves smoothly from its initial position K along the static characteristic of the resistance (the full curve in Fig. 82*a*) since the static characteristics (dashed curves) gradually move to positions *1*, *2*, *3*. On reaching point A_1 corresponding to characteristic *3* the angular velocity increases jumpwise and the working point moves to A_2. From this point the angular velocity will again increase smoothly on transition to characteristic *4* and the working point will continue to glide along the portion A_2L of the resistance characteristic.

If the subsequent adjustment will be made for lower speed, then the working point will glide smoothly along the resistance characteristic over the section LA_2B_1. Having reached point B_1 corresponding

to characteristic *2*, the angular velocity will jump down and the working point will move to B_2. Further the angular velocity will decrease smoothly with smooth transition to characteristic *1* and the working point will glide smoothly along the section B_2K of the resistance characteristic.

The steeper the motor characteristics, the smaller the jumps of the angular velocity described above (see Fig. 82*b*). With sufficiently steep characteristics when within the range of angular velocity adjustment

$$\frac{dM}{d\omega} < \min \frac{dM_{mean}}{d\omega} \tag{11}$$

there will be no angular velocity jumps at all.

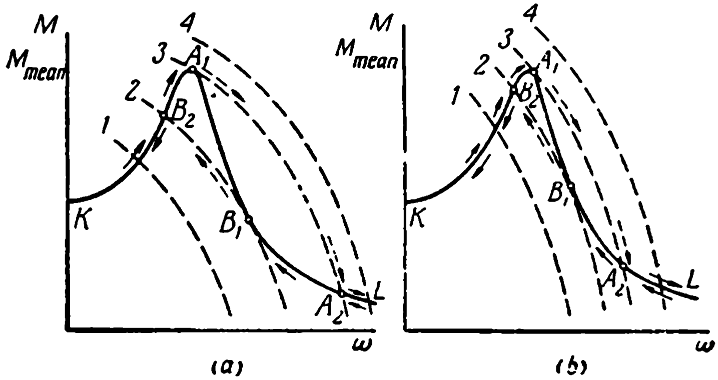


Figure 82

Another nonlinear effect consists in that the rate of increase in vibration range at resonance gradually diminishes. Let us treat the phenomenon schematically, taking, as an instance, the resonant vibrations of a very simple conservative system excited by a sinusoidally varying force (see Fig. 7). In accordance with formula (14) Sec. 7, the solution of the differential equation of the motion takes the following form:

$$x = \frac{F_a}{2m\omega} t \sin \omega t \tag{12}$$

where F_a = amplitude of the sinusoidal exciting force $F_a \cos \omega t$
m = mass of the vibrating body
ω = frequency of resonance vibration.

Let us assume that the static characteristic of the motor is absolutely rigid at $N \leqslant N_0$ where N_0 is the limiting value of the mean power of the energy source. Let us assume also that the frequency ω of the sinusoidal excitation does not vary whatever the rate of increase of the swing of vibrations.

Differentiating expression (12) with respect to time, we obtain

$$\dot{x} = \frac{F_a}{2m\omega}(\sin \omega t + t\omega \cos \omega t) \tag{13}$$

At the moments of time

$$t = \frac{2\pi n}{\omega}, \quad (n = 1, 2, \ldots) \tag{14}$$

the total energy of the system

$$E_n = \frac{F_a^2 \pi^2 n^2}{2m\omega^2} \tag{15}$$

It follows from this expression that the mean power consumed by the system during the time interval between the $(n-1)$ and the $(n+1)$ cycles will be

$$N_{mean} = \frac{\omega}{4\pi}(E_{n+1} - E_{n-1}) = \frac{F_a^2 \pi n}{2m\omega} \tag{16}$$

i. e., with $F_a = \text{const}$ the power increases without bound. Equality (16) holds at $N_{mean} \leqslant N_0$, i.e., up to the n_1 cycle defined by the relation [1]

$$n_1 = \left[\frac{2m\omega N_0}{\pi F_a^2}\right] + 1 \tag{17}$$

which corresponds to the moment of time

$$t_1 = \frac{4mN_0}{F_a^2} \tag{18}$$

in accordance with equality (14).

With $t > t_1$ the "amplitude" of the exciting force starts decreasing since with the conditions assumed this is the only way to maintain the power balance. Denoting the varying "amplitude" by F_*, we can write the condition of the power balance [2] as follows:

$$N_0 = \frac{F_*^2 \pi n}{2m\omega}, \quad (n \geqslant n_1) \tag{19}$$

Hence

$$F_* = \sqrt{\frac{2m\omega N_0}{\pi n}} \tag{20}$$

or, taking into account expression (14),

$$F_* = 2\sqrt{\frac{mN_0}{t}}, \quad (t \geqslant t_1) \tag{21}$$

---

[1] The notation $[A]$ is used for the maximum integer not exceeding A.

[2] It is presumed that F_* changes sufficiently slowly.

The equations of the vibration envelope are

$$\left.\begin{aligned} X_1 &= \pm \frac{F_a t}{2m\omega}, \quad (t \leqslant t_1) \\ X_2 &= \pm \frac{F_* t}{2m\omega} = \pm \sqrt{\frac{N_0 t}{c}}, \quad (t \geqslant t_1) \end{aligned}\right\} \tag{22}$$

where c is the spring stiffness.

Consequently, with $t \geqslant t_1$ the swing of resonant vibrations increases not according to the linear law, as it must be in linear systems without damping, but according to the parabolic law, i.e., the rate of increase is much lower. This is illustrated in Fig. 83.

We have discussed the simplest case of nonstationary vibrations. However the qualitative result remains the same with other motor characteristics and in more complicated systems.

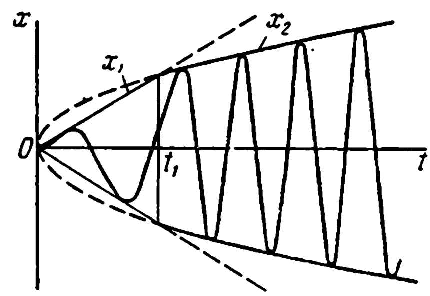


Figure 83

The static characteristics must not be made use of with those nonstationary processes which are characterized by rather rapid changes in the angular velocity of rotation of unbalanced masses. The behaviour of the vibrating system is described by nonreduced differential equations, for example, by Eqs. (1), Sec. 34, for the system shown in Fig. 76 where M is the variable motor moment defined by differential or integro-differential equations which describe the dynamic processes occurring in motors. These processes may be electrodynamic, hydrodynamic, etc., depending on the type of motor. If all the variables determining the electric, hydraulic or other processes in the motor are eliminated from the equations, the following relation will result:

$$f\left(M, \varphi, \frac{d\varphi}{dt}, \frac{d^2\varphi}{dt^2}, \ldots, \frac{d^n\varphi}{dt^n}\right) \tag{23}$$

In some cases the following explicit form can be derived from (23):

$$M = M\left(\varphi, \frac{d\varphi}{dt}, \frac{d^2\varphi}{dt^2}, \ldots, \frac{d^n\varphi}{dt^n}\right) \tag{24}$$

With a nonstationary process where the control of the motor follows a prescribed program (i.e., a program partly or completely independent of the behaviour of the vibrating system and of the processes occurring in the motor system), Eq. (24) becomes nonautonomous:

$$M = M\left(t, \varphi, \frac{d\varphi}{dt}, \frac{d^2\varphi}{dt^2}, \ldots, \frac{d^n\varphi}{dt^n}\right) \tag{25}$$

With Eq. (24) taken into account the differential equations (1), Sec. 34, take the following form:

$$\left.\begin{aligned} &(m_1+m_0)\ddot{x}+b\dot{x}+cx=m_0r(\ddot{\varphi}\sin\varphi+\dot{\varphi}^2\cos\varphi)\\ &J\ddot{\varphi}-m_0r\ddot{x}\sin\varphi=M(\varphi,\dot{\varphi},\ddot{\varphi},\ldots,\varphi^{(n)}) \end{aligned}\right\} \tag{26}$$

Thus the study of transient (as well as steady-state) processes characterized by considerable values of turning angle derivatives with respect to time requires integration of a more complicated set of equations whose order is, generally speaking, higher than that of the initial set of equations (1), Sec. 34. The integration is usually carried out by computers, less often by using numerical and approximate methods.

In some cases expression (24) may be simplified. Thus, for induction and commutator a.c. motors, d.c. motors, steam turbines and other drives one may, in a number of cases, assume that

$$M=M(\dot{\varphi},\ddot{\varphi}) \tag{27}$$

over a wide range of $\dot{\varphi}$ and $\ddot{\varphi}$ values and

$$M=M'-k_1\dot{\varphi}+k_2\ddot{\varphi} \tag{28}$$

in the neighbourhood of the working point.

Of considerable practical interest is the calculation of the minimum starting moment of the motor sufficient to start a centrifugal vibration machine. We shall consider the beginning of the start ensured if the motor gives the unbalanced mass half a turn from the lower position, the unbalanced mass acquiring the angular velocity $\dot{\varphi}>0$. Assuming the motor torque M_{mot} and the moment of the resistance force M_{res} to be constant and neglecting at the beginning of the start the displacement of the generator body (the displacement of the body facilitates starting), we can write the following differential equation of the starting process:

$$J\ddot{\varphi}+m_0gr\sin\varphi=M_{mot}-M_{res},\quad (M_{mot}-M_{res}>0) \tag{29}$$

where φ, m_0 and r have the same meaning as in discussing the arrangement in Fig. 106, and g is the acceleration of the freely falling body.

The first integral of the equation

$$\frac{\dot{\varphi}^2}{2}=\frac{\dot{\varphi}_0^2}{2}+\frac{m_0gr}{J}(\cos\varphi-\cos\varphi_0)+\frac{M_{mot}-M_{res}}{J}(\varphi-\varphi_0) \tag{30}$$

Suppose that the start takes place with the unbalanced mass at the lower position when $\varphi_0=0$. Then with a sufficiently small difference $M_{mot}-M_{res}$ the initial increase of the angular velocity $\dot{\varphi}$ will

be succeeded by a decrease in it due to the moment of the weight of the unbalanced mass exceeding the difference $M_{mot} - M_{res}$. In order to ensure the beginning of the start it is necessary that at the end of the decreasing of the angular velocity its minimum value be

$$\dot{\varphi}_{min} > 0 \tag{31}$$

When the angular velocity $\dot{\varphi}$ is at a minimum, the angular acceleration $\ddot{\varphi}$ is zero and we thus obtain from Eq. (29)

$$\sin \varphi_m = \frac{M_{mot} - M_{res}}{m_0 g r} \tag{32}$$

where φ_m is the turning angle of the unbalanced mass at which $\dot{\varphi}_{min}$ is reached; this angle must lie within the interval $\pi/2 < < \varphi_m < \pi$ because the condition for the minimum of $\dot{\varphi}$ is that the inequality

$$\dddot{\varphi}_m = -\frac{m_0 g r}{J} \dot{\varphi}_{min} \cos \varphi_m > 0 \tag{33}$$

be satisfied.

Substituting into the integral (30) the initial values $\varphi_0 = 0$, $\dot{\varphi}_0 = 0$, we obtain

$$\varphi_m = \frac{2 m_0 g r}{M_{mot} - M_{res}} \sin^2 \frac{\varphi_m}{2} \tag{34}$$

or, in view of equality (32),

$$\varphi_m = \tan \frac{\varphi_m}{2} \tag{35}$$

The required solution of Eq. (35) is

$$\varphi_m = 2.33 \tag{36}$$

Taking account of inequality (31), we now obtain from expression (32)

$$M_{mot} > M_{res} + m_0 g r \sin \varphi_m \tag{37}$$

and since $\sin \varphi_m = 0.725$

$$M_{mot} > M_{res} + 0.725 m_0 g r \tag{38}$$

If the quantity M_{mot} furnished by this inequality is considerably greater than the starting moment of the motor capable of sustaining the steady-state vibrations of the machine, the starting can be facilitated by initiating it with the unbalanced masses in the upper position or letting the masses drop from a position near the upper one and switching on the motor

when the unbalanced mass reaches the lower position, when under the action of the gravity moment the mass has acquired a certain angular velocity.

In the latter case the initial conditions at starting will be

$$\varphi_0=0, \quad \dot{\varphi}_0=\sqrt{\frac{2m_0gr}{J}-\frac{\pi M_{res}}{J}}$$

On inserting these conditions into the integral (30) and making use of equality (35) we obtain the transcendental equation

$$\varphi_m+\cot\frac{\varphi_m}{2}=\frac{\pi M_{res}}{M_{mot}-M_{res}} \tag{39}$$

Upon solving this equation for φ_m we can use relation (37) to determine M_{mot}. This value of the starting torque may prove perceptibly less than that calculated from inequality (38).

The easiest start with the unbalanced masses in the upper position is achieved when φ_m is determined by the equation

$$\varphi_m+\cot\frac{\varphi_m}{2}=\pi \tag{40}$$

whose root, corresponding to the minimum angular velocity, is

$$\varphi_m=\pi \tag{41}$$

Hence, in accordance with inequality (37), the starting condition takes the form

$$M_{mot}>M_{res} \tag{42}$$

36. Self-Synchronization and Vibratory Sustaining of Rotation

The vibrations of working members in many vibration machines are excited by several, mostly by two, concurrently operating centrifugal vibration generators (unbalanced masses); see, for example, the diagrams in Fig. 58 (*d-h*, *j-l*, *n*, *p*). In order to ensure proper operation of the machine the rotations of the vibration generators must be concurrent, i.e., certain prescribed ratios between their mean velocities and phases must be maintained (usually the equality of the moduli of mean angular velocities and the in-phase or antiphase rotation are required). The characteristics of vibration generators are not strictly identical in practice. Therefore, in the general case, the frequencies at which individual vibration generators rigidly mounted on a fixed foundation would rotate are different.

Concurrent rotation of the vibration generators mounted on a common working member can be achieved by positive synchronization, kinematic or electric. Under certain conditions the centrifugal vibration generators actuated by drives of the non-synchronous type operate synchronously in spite of the differences in the para-

meters and the absence of direct connections between the drives. The vibrations of the working member (the carrier body) serve as the channel through which the energy is redistributed between the vibration generators, this exchange leading to their concurrent rotation usually called *self-synchronization*. The use of this phenomenon allows considerable simplification of the machine design.

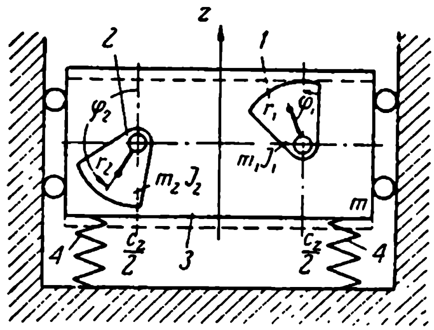


Figure 84

Self-synchronization cannot be studied within the framework of the linear theory by assuming that vibration generators rotate uniformly and synchronously. It belongs to the category of typically nonlinear phenomena akin to entrainment (cf. Section 21). This phenomenon occurs not only in concurrently operating vibration generators, but also in many other dynamic systems having close vibration frequencies or rotation speeds, provided that certain constraints, sometimes very weak ones, have been imposed on the systems. The general theory of synchronization has been developed in the works of I. Blekhman and other investigators.

We shall restrict our treatment to a particular example which, though very simple, retains the characteristic features of the general self-synchronization problem.

Figure 84 is the dynamic model of a vibration machine with two vibration generators *1* and *2*; the working member *3* of mass m_* has one degree of freedom and is supported by ideal springs *4* whose total stiffness is c_z. Let m_k, r_k and J_k denote the mass, its eccentricity and the moment of inertia of the kth unbalance ($k = 1, 2$). Let us denote by z the displacement of the working member measured from the equilibrium position, and by φ_k the turning angle of the kth unbalanced mass measured counterclockwise from the positive direction of the z-axis. We introduce the number ε_k equal to $+1$ if the kth unbalanced mass rotates in the positive sense and to -1 in the opposite case. The coefficient of viscous resistance to the rotation of the kth unbalanced mass is denoted by b_k and the motor torque is given by the straight-line portion of the static characteristic of the motor $\varepsilon_k L_k - b_k^* \dot{\varphi}_k$ where L_k and b_k^* are constants.

We consider only the nonresonance case, neglecting accordingly the dissipative resistance to the motion of the body. The moment of the weight of unbalanced masses is also neglected as it is irrelevant to the discussion. The equations of motion can now be written

in the following form:

$$\left.\begin{aligned}
&m\frac{d^2z}{dt^2}+c_zz=m_1r_1\left[\left(\frac{d\varphi_1}{dt}\right)^2\cos\varphi_1+\frac{d^2\varphi_1}{dt^2}\sin\varphi_1\right]+\\
&+m_2r_2\left[\left(\frac{d\varphi_2}{dt}\right)^2\cos\varphi_2+\frac{d^2\varphi_2}{dt^2}\sin\varphi_2\right]\\
&J_k\frac{d^2\varphi_k}{dt^2}+b_k\frac{d\varphi_k}{dt}=\varepsilon_kL_k-b_k^*\frac{d\varphi_k}{dt}+\\
&+m_kr_k\frac{d^2z_k}{dt^2}\sin\varphi_k,\quad (k=1,\ 2)
\end{aligned}\right\}\tag{1}$$

where $m=m_*+m_1+m_2$.

The task is to find the solution of the set of equations (1) in the form

$$z=z(\omega t),\quad \varphi_k=\varepsilon_k[\omega t+g_k(\omega t)],\quad (k=1,\ 2)\tag{2}$$

where z and g_k = periodic functions of their argument ωt with period 2π

ω = unknown synchronous frequency which is also to be determined.

The terms entering into Eqs. (1) differ by order of magnitude. This may be directly shown by choosing a suitable scale for the variables. Let us denote

$$\zeta=\frac{z}{H},\quad \tau=\omega t\tag{3}$$

where H has the dimension of length. For the sake of brevity, the quantities ζ and τ will be called the dimensionless displacement and the dimensionless time, respectively. It is convenient to introduce as new variables, instead of the phases φ_1 and φ_2, their combinations

$$\varphi=\frac{1}{2}(\varepsilon_1\varphi_1+\varepsilon_2\varphi_2),\quad \psi=\frac{1}{2}(\varepsilon_1\varphi_1-\varepsilon_2\varphi_2)\tag{4}$$

If we also introduce the notations

$$\left.\begin{aligned}
&H=\frac{m_1r_1+m_2r_2}{m},\quad \mu=2\frac{(m_1r_1)^2+(m_2r_2)^2}{m(J_1+J_2)},\quad \gamma_1^2=\frac{c_z}{m\omega_0^2}\\
&\omega_0=\frac{1}{2}(\omega_1+\omega_2),\quad \omega_k=\frac{L_k}{b+b_k^*}=\frac{\omega_0}{1+\mu p_k}\\
&\omega=\frac{\omega_0}{1+\mu k};\quad p_1=-p_2=p\\
&\lambda_k=\frac{L}{J_k\omega_0^2}=\lambda(1+\mu r_k),\quad \lambda=\frac{1}{2\omega_0^2}\left(\frac{L_1}{J_1}+\frac{L_2}{J_2}\right),\\
&r_1=-r_2=r,\quad (k=1,\ 2)
\end{aligned}\right\}\tag{5}$$

then Eqs. (1) will become

$$\left.\begin{aligned}\ddot{\zeta}+\gamma_1^2\zeta=[(\dot{\varphi}^2+\dot{\psi}^2)\cos\varphi+\ddot{\varphi}\sin\varphi]\cos\psi+\\+(\ddot{\psi}\cos\varphi-2\dot{\varphi}\dot{\psi}\sin\varphi)\sin\psi\\\ddot{\varphi}+\lambda\dot{\varphi}=\lambda+\mu\,[\ddot{\zeta}\sin\varphi\cos\psi+\\+2\lambda h-\lambda h\dot{\varphi}-\lambda(r+p)\,\dot{\psi}]\\\ddot{\psi}+\lambda\dot{\psi}=\mu\,[\ddot{\zeta}\cos\varphi\sin\psi+\\+\lambda r-\lambda(r+p)\,\dot{\varphi}-\lambda h\dot{\psi}]\end{aligned}\right\}\tag{6}$$

The dots are used to denote differentiation with respect to τ.

Note that the terms containing the dimensionless acceleration of the working member $\ddot{\zeta}$ enter into the second and third equations only with the factor μ. Since it is these terms that determine the coupling of the motion of the working member and the rotation of the unbalanced masses, one may say that the factor μ is the measure of the strength of this coupling. In vibration machines of the type considered the factor μ is small as compared to unity and therefore we may use the small-parameter method to solve the self-synchronization problem. Note that the parameter μ is similar to the quantity α^2 defined by formula (18), Sec. 33.

The smallness of μ is reflected also in notations (5). Assuming the vibration generators to be almost identical, we express the differences in the parameters in the form of terms having the order of μ. The selection of the unknown synchronous frequency ω as time scale has led to the absence in explicit form of the frequency in the equations of motion. The problem is thus reduced to constructing the solution of the set of equations (6) in the form

$$\zeta=\zeta(\tau),\quad \varphi=\tau+g(\tau),\quad \psi=\psi(\tau)\tag{7}$$

and determining the correction μh to the frequency.

In applying the small-parameter method (see Section 19) we shall seek the solution in the form of power series in μ, i.e.,

$$\left.\begin{aligned}\varphi(\tau)=\varphi^{(0)}(\tau)+\mu\varphi^{(1)}(\tau)+\dots\\\psi(\tau)=\psi^{(0)}(\tau)+\mu\psi^{(1)}(\tau)+\dots\end{aligned}\right\}\tag{8}$$

and it will be sufficient for our purpose, as can be readily seen from what follows, to be contented with the zero approximation to ζ. So we shall everywhere write $\zeta(\tau)$ instead of $\zeta^{(0)}(\tau)$. The equations for the zero approximation to φ and ψ can be derived by setting

$\mu = 0$ in the second and third of Eqs. (6):

$$\left.\begin{aligned}\ddot{\varphi}^{(0)} + \lambda\dot{\varphi}^{(0)} = \lambda\\ \ddot{\psi}^{(0)} + \lambda\dot{\psi}^{(0)} = 0\end{aligned}\right\} \tag{9}$$

Thus the equations in φ and ψ for the zero approximation prove to be independent and not related to the first of Eqs. (6). These equations are satisfied by a solution of the form

$$\varphi^{(0)}(\tau) = \tau + \varphi_0^{(0)}, \quad \psi^{(0)}(\tau) = \psi_0^{(0)} \tag{10}$$

where $\varphi_0^{(0)}$ and $\psi_0^{(0)}$ are constants.

It is clear that this solution also satisfies conditions (7) simultaneously, since the constant may be regarded as a particular case of the periodic function.

Substituting solution (10) into the first of Eqs. (6), we obtain

$$\ddot{\zeta} + \gamma_1^2\zeta = \cos\psi_0^{(0)}\cos(\tau + \varphi_0^{(0)}) \tag{11}$$

i.e., the well-known equation of vibrations of a single-degree-of-freedom system.

The zero approximation solution (10) depends on two arbitrary constants. One of them, viz. $\varphi_0^{(0)}$, can be made equal to zero by a suitable choice of the origin of the time. Accordingly we shall put $\varphi_0^{(0)} = 0$ in the following discussion. As to the second constant $\psi_0^{(0)}$ which is equal, by formula (4), to half the phase difference between the vibration generators, it remains indeterminate.

In order to determine $\psi_0^{(0)}$ and h we now turn to the first-order terms in the expansions (8). We insert the expansions into the second and third of Eqs. (6) and replace $\varphi^{(0)}(\tau)$ and $\psi^{(0)}(\tau)$ by expressions (10) found earlier. The zero-order terms cancel out in accordance with Eqs. (9); rejecting the terms of the second and higher orders in μ, we obtain the equations of the first approximation

$$\left.\begin{aligned}\ddot{\varphi}^{(1)} + \lambda\dot{\varphi}^{(1)} = \ddot{\zeta}(\tau)\sin\tau\cos\psi_0^{(0)} + \lambda h\\ \ddot{\psi}^{(1)} + \lambda\dot{\psi}^{(1)} = \ddot{\zeta}(\tau)\cos\tau\sin\psi_0^{(0)} - \lambda p\end{aligned}\right\} \tag{12}$$

where, according to Eq. (11),

$$\ddot{\zeta} = \frac{\cos\psi_0^{(0)}}{1-\gamma_1^2}\cos\tau \tag{13}$$

It can be seen now that the right-hand sides of Eqs. (12) are known functions of τ.

Let us introduce the notations

$$\left.\begin{aligned}\ddot{\zeta}(\tau)\sin\tau\cos\psi_0^{(0)} \equiv \Phi(\tau, \psi_0^{(0)})\\ \ddot{\zeta}(\tau)\cos\tau\sin\psi_0^{(0)} - \lambda p \equiv \Psi(\tau, \psi_0^{(0)})\end{aligned}\right\} \tag{14}$$

The first integrals of Eqs. (12) can be written in the form

$$\left.\begin{aligned}\dot{\varphi}^{(1)}(\tau) &= e^{-\lambda\tau}\left[\int_0^\tau \Phi(\theta,\ \psi_0^{(0)})\, e^{\lambda\theta}\, d\theta + C_1\right] + h\\ \dot{\psi}^{(1)}(\tau) &= e^{-\lambda\tau}\left[\int_0^\tau \Psi(\theta,\ \psi_0^{(0)})\, e^{\lambda\theta}\, d\theta + C_2\right]\end{aligned}\right\} \tag{15}$$

The second integration with the use of integration by parts leads to the following form of the general solution of the first approximation equations:

$$\left.\begin{aligned}\varphi^{(1)}(\tau) &= C_3 + h\tau + \frac{C_1}{\lambda}(1 - e^{-\lambda\tau}) -\\ &- \frac{1}{\lambda} e^{-\lambda\tau}\int_0^\tau \Phi(\theta,\ \psi_0^{(0)})\, e^{\lambda\theta}\, d\theta + \frac{1}{\lambda}\int_0^\tau \Phi(\theta,\ \psi_0^{(0)})\, d\theta\\ \psi^{(1)}(\tau) &= C_4 + \frac{C_2}{\lambda}(1 - e^{-\lambda\tau}) -\\ &- \frac{1}{\lambda} e^{-\lambda\tau}\int_0^\tau \Psi(\theta,\ \psi_0^{(0)})\, e^{\lambda\theta}\, d\theta + \frac{1}{\lambda}\int_0^\tau \Psi(\theta,\ \psi_0^{(0)})\, d\theta\end{aligned}\right\} \tag{16}$$

The quantities C_1, C_2, C_3, C_4 in Eqs. (15) and (16) are constants whose numerical values ase insignificant. Expansions (8) must satisfy certain conditions in accordance with equalities (7), viz.: $\varphi(\tau) - \tau$ and $\psi(\tau)$ must be periodic functions with period 2π. Since zero approximation (10) satisfies these conditions, the functions $\varphi^{(1)}(\tau)$ and $\psi^{(1)}(\tau)$ must be periodic:

$$\left.\begin{aligned}\varphi^{(1)}(2\pi) &= \varphi^{(1)}(0)\\ \psi^{(1)}(2\pi) &= \psi^{(1)}(0)\end{aligned}\right\} \tag{17}$$

Of course, the same conditions are imposed on the derivatives $\dot{\varphi}^{(1)}$ and $\dot{\psi}^{(1)}$.

Substituting solutions (16) and (15) in the periodicity conditions, we see that the phase difference $\psi_0^{(0)}$ and the correction h for the frequency must be determined from the relations

$$P_1(\psi_0^{(0)}) \equiv \frac{1}{\lambda}\int_0^{2\pi} \Psi(\tau,\ \psi_0^{(0)})\, d\tau = 0 \tag{18}$$

$$P_2(\psi_0^{(0)}) \equiv \frac{1}{\lambda}\int_0^{2\pi} \Phi(\tau,\ \psi_0^{(0)})\, d\tau = -2\pi h \tag{19}$$

In the vibration theory, equations of the type (18) are called *determining* or *bifurcation equations*. Equation (19) represents the power balance to the first approximation. Since the dissipative resistance to the rotation of the vibration generator is, in principle, included in the motor characteristic and the dissipative resistance to the motion of the body is neglected, the active power must be zero. Hence $h = 0$. The same result is obtained by direct calculation. In fact, as it follows from formulas (13) and (14), the integral contained in Eq. (19)

$$\int_0^{2\pi} \Phi(\tau,\ \psi_0^{(0)})\, d\tau = \frac{\cos^2 \psi_0^{(0)}}{1-\gamma_1^2} \int_0^{2\pi} \sin\tau\cos\tau\, d\tau = 0$$

Thus, the correction for the first frequency approximation proves to be zero.

Substituting the expression for $\Psi(\tau,\ \psi_0^{(0)})$ defined by identity (14) into formula (18), we obtain the equation for $\psi_0^{(0)}$:

$$P_1(\psi_0^{(0)}) = \frac{\pi}{\lambda}\left(\frac{\sin\psi_0^{(0)}\cos\psi_0^{(0)}}{1-\gamma_1^2} - 2\lambda p\right) = 0 \qquad (20)$$

If, at a first approximation, the parameters of the vibration generators are the same, i.e., $p = 0$, then Eq. (20) has two solutions, $\psi_*^{(0)} = 0$ and $\psi_*^{(0)} = \pi/2$. The first of the solutions corresponds to the in-phase and the second to the antiphase rotation of the vibration generators. With sufficiently small p we obtain two solutions close to 0 and $\pi/2$, respectively.

All the conditions of the problem are formally satisfied by both solutions. However, only one of them is actually realized, the one that is stable with respect to small perturbations inevitably present in any real system.

Let us consider the function $\lambda\psi(\tau) + \dot{\psi}(\tau)$ written accurate within first-order terms in μ. In accordance with expressions (10), (15) and (16) we have

$$\lambda\psi(\tau) + \dot{\psi}(\tau) = \lambda\psi_0^{(0)} + \mu(\lambda C_1 + C_2) + \mu\int_0^{\tau} \Psi(\theta,\ \psi_0^{(0)})\, d\theta \qquad (21)$$

(one could also consider the function $\psi(\tau)$ but this involves a lengthy derivation). Let $\psi_*^{(0)} = \psi_*^{(0)} + \Delta\psi_*$ where $\psi_*^{(0)}$ is one of the solutions of the bifurcation equation (18), $\Delta\psi_*$ a small perturbation; the function (21) can now be presented accurate to first-order terms in $\Delta\psi_*$ in the following form:

$$\lambda\psi(\tau) + \dot{\psi}(\tau) = \lambda\psi_*^{(0)} + \lambda\Delta\psi_* + \mu(\lambda C_1 + C_2) +$$

$$+ \mu\int_0^{\tau} \Psi(0,\ \psi_*^{(0)})\, d\theta + \mu\Delta\psi_* \frac{\partial}{\partial\psi_*^{(0)}} \int_0^{\tau} \Psi(\theta,\ \psi_*^{(0)})\, d\theta \qquad (22)$$

Taking into account condition (18), we can write

$$\left.\begin{aligned}\lambda\psi(0)+\dot{\psi}(0)&=\lambda\psi_*^{(0)}+\lambda\Delta\psi_*+\mu(\lambda C_1+C_2)\\ \lambda\psi(2\pi)+\dot{\psi}(2\pi)&=\lambda\psi_*^{(0)}+\lambda\left(1+\mu\frac{dP_1}{d\psi_*^{(0)}}\right)\Delta\psi_*+\mu(\lambda C_1+C_2)\end{aligned}\right\}\qquad(23)$$

Comparing the last two expressions, we see that if the perturbation at the initial moment is equal to $\Delta\psi_*$, it becomes $\left(1+\mu\frac{dP_1}{d\psi_*^{(0)}}\right)\Delta\psi_*$ at $\tau=2\pi$. If the number in parentheses is less than unity, one may conclude that the perturbation in phase caused in the system decreases with time and consequently the synchronous motion corresponding to the phase $\psi_*^{(0)}$ is stable. Otherwise, the perturbation increases and the corresponding synchronous motion is unstable.

Thus, the stability condition of a synchronous motion corresponding to the phase $\psi_0^{(0)}$ takes the form

$$\frac{dP_1}{d\psi_*^{(0)}}<0\qquad(24)$$

Inserting into (24) the left-hand side of Eq. (20), we obtain

$$\frac{1}{1-\gamma_1^2}\cos 2\psi_*^{(0)}<0\qquad(25)$$

Hence we conclude that the synchronous in-phase motion ($\psi_*^{(0)}=0$) is stable at $\gamma_1>1$, i.e., in the preresonance region, and that the synchronous antiphase motion is stable at $\gamma_1<1$, i.e., in the postresonance region. These conclusions are valid only for γ_1 values sufficiently different from unity when the system may be considered non-resonant.

The self-synchronization problem for n vibration generators mounted on a system of carrier bodies having f degrees of freedom can be solved by a similar procedure. If we introduce as the coordinates describing the rotation of the unbalanced masses the functions

$$\varphi=\frac{1}{n}\sum_{s=1}^{n}\varepsilon_s\varphi_s,\quad \psi_j=\varepsilon_j\varphi_j-\varphi,\qquad (j=1,\ 2,\ \ldots,\ n-1)\qquad(26)$$

then the quantities $\psi_j^{(0)}$, at a zero approximation, again prove constants and their values corresponding to synchronous motions are found from the following set of transcendental bifurcation equations:

$$P_j(\psi_{1*}^{(0)},\ \psi_{2*}^{(0)},\ \ldots,\ \psi_{n-1,\,*}^{(0)})=0,\qquad(j=1,\ 2,\ \ldots,\ n-1)\qquad(27)$$

and the correction to the frequency is found from the equation

$$P_n(\psi_{1*}^{(0)},\ \psi_{2*}^{(0)},\ \ldots,\ \psi_{n-1,\,*}^{(0)})=-2\pi h\qquad(28)$$

The structure of these equations can be partly elucidated by comparing with formulas (18) and (19). The stability condition of the

synchronous motion corresponding to a certain solution of system (27) is the generalization of condition (24) and can be stated as follows. All the roots of the algebraic equation of degree $(n-1)$

$$\begin{vmatrix} \frac{\partial P_1}{\partial \psi_{1*}^{(0)}} - z & \frac{\partial P_1}{\partial \psi_{2*}^{(0)}} & \dots & \frac{\partial P_1}{\partial \psi_{n-1,*}^{(0)}} \\ \frac{\partial P_2}{\partial \psi_{1*}^{(0)}} & \frac{\partial P_2}{\partial \psi_{2*}^{(0)}} - z & \dots & \frac{\partial P_2}{\partial \psi_{n-1,*}^{(0)}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial P_{n-1}}{\partial \psi_{1*}^{(0)}} & \frac{\partial P_{n-1}}{\partial \psi_{2*}^{(0)}} & \dots & \frac{\partial P_{n-1}}{\partial \psi_{n-1,*}^{(0)}} - z \end{vmatrix} = 0$$

or, in brief notation,

$$\left| \frac{\partial P_j}{\partial \psi_{k*}^{(0)}} - \delta_{jk} z \right| = 0 \tag{29}$$

must have negative real parts.

Thus, in order to solve the self-synchronization problem one must construct the approximate solution of a complete set of equations, taking account of the effect of the vibrations of a carrier body on the rotation of the unbalanced masses and then investigate the stability of the solution. I. Blekhman and B. Lavrov have formulated an integral criterion of the stability of synchronous motions that permits one to simplify considerably the procedure. The applicability of the integral criterion is limited to systems that are conservative at a zero approximation.

The essence of the integral criterion of stability for synchronous motions can be stated briefly as follows. Consider the zero approximation equations for the carrier body. As the above example shows, these equations describe the motion of the so-called auxiliary body[1] under the influence of sinusoidal exciting forces with arbitrary phase differences. It is found that the phase differences to which synchronous motions can formally correspond are determined from the conditions of stationariness of the period-averaged value of the Lagrangian function in the steady motion of the carrier body. Actually only those motions are realized, i.e., are stable, to which corresponds the minimum of this mean value.

A somewhat more complicated example will now be used to illustrate the application of the integral criterion. Consider a dynamic model of a vibration machine whose working member *1* has now not one but three degrees of freedom in a plane motion (Fig. 85). Let x denote the horizontal displacement of the working member and α its turning angle. We assume that, apart from the stiffness c_z, the spring mount *2* has the stiffness c_x in the horizontal direction

[1] The term *auxiliary body* is used to denote a carrier body with the unbalanced masses fixed to it and taken to be lumped on their axes.

and the torsional stiffness c_α. The distance between the rotation axes of the unbalanced masses is denoted by $2a$, and the moment of inertia of the working member by J_*; the rest of the notations are the same as those in the preceding example. The upper index $^{(0)}$ of all the zero approximation quantities will be omitted to simplify the notation since subsequent approximations are not considered at all.

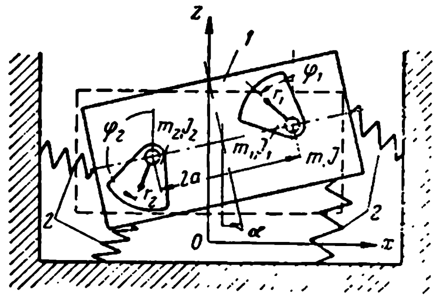


Figure 85

Setting the constant φ_0 equal to zero, which can be achieved by a suitable choice of the initial moment of time, one finds from formulas (4) and (10)

$$\varphi_1=\varepsilon_1(\omega_0 t+\psi_0), \quad \varphi_2=\varepsilon_2(\omega_0 t-\psi_0) \tag{30}$$

The equations of the motion of the auxiliary body take the form

$$\left.\begin{aligned} m\frac{d^2z}{dt^2}+c_z z&=\frac{F_a}{2}[\cos\varepsilon_1(\omega_0t+\psi_0)+\cos\varepsilon_2(\omega_0t-\psi_0)]\\ m\frac{d^2x}{dt^2}+c_x x&=-\frac{F_a}{2}[\sin\varepsilon_1(\omega_0t+\psi_0)+\sin\varepsilon_2(\omega_0t-\psi_0)]\\ J\frac{d^2\alpha}{dt^2}+c_\alpha\alpha&=\frac{aF_a}{2}[\cos\varepsilon_1(\omega_0t+\psi_0)-\cos\varepsilon_2(\omega_0t-\psi_0)] \end{aligned}\right\} \tag{31}$$

where $J=J_*+2a^2m_1$; $F_a=2m_1r_1\omega_0^2$.

For the zero approximation we assume the vibration generators to be identical ($m_1=m_2$, $r_1=r_2$).

Equations (31) can be rewritten as follows:

$$\left.\begin{aligned} m\frac{d^2z}{dt^2}+c_z z&=F_a\cos\psi_0\cos\omega_0t\\ m\frac{d^2x}{dt^2}+c_x x&=-\varepsilon_1F_a\begin{Bmatrix}\cos\psi_0\sin\omega_0t\\ \sin\psi_0\cos\omega_0t\end{Bmatrix}\\ J\frac{d^2\alpha}{dt^2}+c_\alpha\alpha&=-aF_a\sin\psi_0\sin\omega_0t \end{aligned}\right\} \tag{32}$$

Here and in the following derivation the upper element of the column in braces corresponds to the case when $\varepsilon_1\varepsilon_2=+1$, and the lower to $\varepsilon_1\varepsilon_2=-1$, i.e., to the rotation of vibration generators in opposite senses.

The periodic solution of Eqs. (32) may be written as

$$\left.\begin{aligned} z&=\frac{F_a\cos\psi_0}{c_z-m\omega_0^2}\cos\omega_0t, \quad x=-\frac{\varepsilon_1F_a}{c_x-m\omega_0^2}\begin{Bmatrix}\cos\psi_0\sin\omega_0t\\ \sin\psi_0\cos\omega_0t\end{Bmatrix}\\ \alpha&=-\frac{aF_a\sin\psi_0}{c_\alpha-J\omega_0^2}\sin\omega_0t \end{aligned}\right\} \tag{33}$$

The Lagrangian function of the system in question takes the form

$$L_0=\frac{m\left(\frac{dz}{dt}\right)^2}{2}-\frac{c_z z^2}{2}+\frac{m\left(\frac{dx}{dt}\right)^2}{2}-\frac{c_x x^2}{2}+\frac{J\left(\frac{d\alpha}{dt}\right)^2}{2}-\frac{c_\alpha \alpha^2}{2} \quad (34)$$

Substituting solution (33) into (34) and calculating the latter's mean value, we obtain

$$\Lambda_0(\psi_0)=\frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} L_0\,dt=\frac{F_a^2}{2m\omega_0^2}\left(\frac{1}{1-\gamma_1^2}\cos^2\psi_0+\right.$$

$$\left.+\frac{1}{1-\gamma_2^2}\begin{Bmatrix}\cos^2\psi_0\\ \sin^2\psi_0\end{Bmatrix}+\frac{a^2m}{J}\cdot\frac{1}{1-\gamma_3^2}\sin^2\psi_0\right) \quad (35)$$

where the following squared dimensionless natural frequencies have been introduced:

$$\gamma_1^2=\frac{c_z}{m\omega_0^2},\quad \gamma_2^2=\frac{c_x}{m\omega_0^2},\quad \gamma_3^2=\frac{c_\alpha}{J\omega_0^2} \quad (36)$$

The unknown values of the synchronous motion phase are to be determined, in accordance with the integral criterion, from the equation $d\Lambda_0/d\psi_0=0$ or

$$-\frac{F_0}{m\omega_0^2}\left(\frac{1}{1-\gamma_1^2}+\frac{\varepsilon_1\varepsilon_2}{1-\gamma_2^2}-\frac{a^2m}{J}\cdot\frac{1}{1-\gamma_3^2}\right)\sin\psi_0\cos\psi_0=0 \quad (37)$$

The equation has two solutions: $\psi_*=0$ and $\psi_*=\pi/2$, i.e., either in-phase or antiphase rotation is possible. In order to ascertain the stability of the solutions we compute the second derivative $d^2\Lambda_0/d\psi_0^2$ and find the following stability condition:

$$\left(\frac{1}{1-\gamma_1^2}+\frac{\varepsilon_1\varepsilon_2}{1-\gamma_2^2}-\frac{a^2m}{J}\cdot\frac{1}{1-\gamma_3^2}\right)\cos 2\psi_*<0 \quad (38)$$

Thus the stability of the synchronous motion corresponding to either of the solutions of Eq. (37) depends on the sign of the expression in the parentheses: the in-phase motion is stable when the expression is negative and unstable if it is positive; on the contrary, the antiphase rotation is stable if the factor before $\cos 2\psi_*$ in inequality (38) is positive.

With $\gamma_2\to\infty$ and $\gamma_3\to\infty$ we have the case formally corresponding to a system having one degree of freedom and the results obtained are the same as in the preceding example.

The moments of the inertia forces (the so-called synchronizing moments) that cause two or more concurrently operating centrifugal vibration generators to rotate synchronously with definite phase angles between them may become so high that even when the motor

of one of the generators is switched off, the generator will continue to rotate in synchronism with the other vibration generators If in this case the reciprocal action of the generator with the motor switched off on the motion of the carrier body may be neglected, then we have an instance of the sustaining of rotation by vibrations.

In distinction to the self-synchronization problem, the sustaining of rotation by vibrations is a non-autonomous problem. It can be formulated, in particular, as follows. Let the axis A of the unbalance in the plane scheme in Fig. 86 perform a prescribed (i.e., independent of the rotation of the unbalanced mass) periodic motion that may be either vibration or a circulating motion. It is required to ascertain whether it is possible to sustain the rotation of the unbalanced mass at a mean angular speed, which is equal to the angular frequency of the exciting motion of its axis, and what is the mutual phasing of its rotation and the exciting motion of the axis.

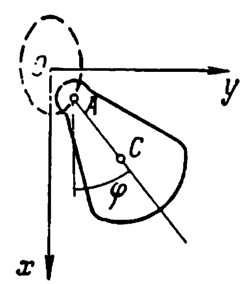


Figure 86

Consider a simple case in which the axis of the unbalanced mass describes an elliptic path whose equations in parametric form are

$$x = a \cos \omega t, \quad y = b \sin \omega t, \quad (b \leqslant a) \tag{39}$$

These are the equations of an ellipse whose coordinate origin is at its centre. The position of the unbalanced mass will be defined by the angle φ between the positive direction of the x-axis and the radius-vector $r = AC$ of its centre of gravity C with respect to the rotation axis A.

The kinetic energy of the unbalance

$$T = \frac{1}{2} m_0 \left[\left(a\omega \sin \omega t + r \frac{d\varphi}{dt} \sin \varphi \right)^2 + \right.$$
$$\left. + \left(b\omega \cos \omega t + r \frac{d\varphi}{dt} \cos \varphi \right)^2 \right] + \frac{1}{2} J_0 \left(\frac{d\varphi}{dt} \right)^2 \tag{40}$$

where m_0 and J_0 are respectively the unbalanced mass and its central moment of inertia.

Let the moment of the resistance to the rotation of the unbalanced mass be $-k\,(d\varphi/dt)$ where k is the coefficient of resistance. Introducing the notation $J = J_0 + m_0 r^2$, we obtain the following differential equation of motion of the unbalanced mass:

$$J \frac{d^2\varphi}{dt^2} + k \frac{d\varphi}{dt} + m_0 r \omega^2 (a \sin \varphi \cos \omega t - b \cos \varphi \sin \omega t) = 0 \tag{41}$$

or in dimensionless quantities

$$\ddot{\varphi} + \lambda \dot{\varphi} + \mu (\sin \varphi \cos \tau - \varepsilon \cos \varphi \sin \tau) = 0 \tag{42}$$

where the dots over φ denote differentiation with respect to τ and

$$\tau = \omega t, \quad \mu = \frac{m_0 ra}{J}, \quad \varepsilon = \frac{b}{a}, \quad \lambda = \frac{k}{J\omega} \tag{43}$$

We shall seek the solution of Eq. (42) representing the synchronous rotation of the unbalanced mass in the direction of the circulating motion of its axis at which $(d\varphi/dt)_{mean} = \omega$. We set accordingly

$$\varphi = \tau + \psi(\tau) \tag{44}$$

where $\psi(\tau)$ is a periodic function whose frequency is equal to or a multiple of unity.

Equation (42) now takes the form

$$\ddot{\psi} + \lambda\dot{\psi} + \mu\,[\sin(\tau + \psi)\cos\tau - \varepsilon\cos(\tau + \psi)\sin\tau] + \lambda = 0 \tag{45}$$

Considering the quantity μ to be a small parameter, we shall represent the solution of the equation by a series in powers of μ:

$$\psi(\tau) = \psi^{(0)}(\tau) + \mu\psi^{(1)}(\tau) + \mu^2\psi^{(2)}(\tau) + \ldots \tag{46}$$

In what follows it is sufficient to limit ourselves to the first two terms of expansion (46). Inserting $\psi = \psi^{(0)} + \mu\psi^{(1)}$ into Eq. (45) and again limiting ourselves to the first-order terms, we obtain after a simple transformation

$$\ddot{\psi}^{(0)} + \lambda\dot{\psi}^{(0)} + \mu\left[\ddot{\psi}^{(1)} + \lambda\dot{\psi}^{(1)} + \frac{1-\varepsilon}{2}\sin(2\tau + \psi^{(0)}) + \right.$$
$$\left. + \frac{1+\varepsilon}{2}\sin\psi^{(0)}\right] + \lambda = 0 \tag{47}$$

The zero approximation equation is derived by rejecting the term proportional to μ. In accordance with the condition assumed earlier $\psi^{(0)}$ is a periodic function. Hence it may be readily seen that the constant term λ must belong to the first approximation equation. Thus

$$\ddot{\psi}^{(0)} + \lambda\dot{\psi}^{(0)} = 0 \tag{48}$$

The periodic solution of Eq. (48) can be only the constant

$$\psi^{(0)} = \psi_0 = \text{const} \tag{49}$$

whose value in this approximation remains arbitrary. To determine it we consider the next approximation. According to expression (47) the first approximation equation becomes

$$\ddot{\psi}^{(1)} + \lambda\dot{\psi}^{(1)} = \Phi(\tau, \psi_0) \tag{50}$$

where we have introduced the notation

$$\Phi(\tau, \psi_0) = -\frac{1-\varepsilon}{2}\sin(2\tau+\psi_0) - \frac{1+\varepsilon}{2}\sin\psi_0 - \frac{\lambda}{\mu} \tag{51}$$

The results obtained in this section can now be made use of. The periodicity condition of the function $\psi^{(1)}$ may be written similarly to relation (18) in the form

$$P(\psi_0) \equiv \frac{1}{\lambda}\int_0^{2\pi} \Phi(\tau, \psi_0)\,d\tau = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{\mu} + \frac{1+\varepsilon}{2}\sin\psi_0\right) = 0 \tag{52}$$

Thus the phase ψ_0 for the steady motion is determined from the equation

$$\sin\psi_0 = -\frac{2\lambda}{\mu(1+\varepsilon)} \tag{53}$$

Upon changing to dimensional notations of the quantities we find that the necessary condition for the rotation to be maintained by vibration is to satisfy the inequality

$$\frac{2k}{(1+\varepsilon)\,\omega m_0 ra} < 1 \tag{54}$$

We denote the solution of Eq. (53) by ψ_*; the periodic solution of Eq. (50) accurate within first-order terms in μ takes the form

$$\psi^{(1)}(\tau) = \frac{1-\varepsilon}{8}\sin(2\tau+\psi_*) \tag{55}$$

and the required solution (46) accurate to first-order terms may be written as follows:

$$\psi(\tau) = \psi_* + \mu\frac{1-\varepsilon}{8}\sin(2\tau+\psi_*) \tag{56}$$

Equation (53) has two essentially different solutions

$$\psi_{*1} = -\sin^{-1}\frac{2\lambda}{\mu(1+\varepsilon)}, \quad \psi_{*2} = \pi + \sin^{-1}\frac{2\lambda}{\mu(1+\varepsilon)} \tag{57}$$

There are solutions of Eq. (45) corresponding to each ψ_* value but only one of the solutions proves stable. In order to ascertain the stability conditions let us assume that at the moment $\tau = 0$ the phase ψ_* acquires a small increment $\Delta\psi_*$, i.e., $\psi_0 = \psi_* + \Delta\psi_*$.

In a way similar to that used in studying the self-synchronization conditions let us now consider the function $\lambda\psi + \dot{\psi}$. In accordance with expression (22) this function, accurate to a constant term

that does not signify, may be written in the following form:

$$\lambda\psi + \dot{\psi} = \lambda\psi_* + \lambda\Delta\psi_* + \mu \int_0^\tau \Phi(\theta, \psi_*)\, d\theta +$$
$$+ \mu\Delta\psi_* \frac{\partial}{\partial\psi_*} \int_0^\tau \Phi(\theta, \psi_*)\, d\theta \tag{58}$$

Hence

$$\left.\begin{aligned} \lambda\psi(0) + \dot{\psi}(0) &= \lambda\psi_* + \lambda\Delta\psi_* \\ \lambda\psi(2\pi) + \dot{\psi}(2\pi) &= \lambda\psi_* + \lambda\left(1 + \mu\frac{dP}{d\psi_*}\right)\Delta\psi_* \end{aligned}\right\} \tag{59}$$

Consequently, if the perturbation was $\Delta\psi_*$ at $\tau = 0$, it becomes equal to $\left(1 + \mu\frac{dP}{d\psi_*}\right)\Delta\psi_*$ at $\tau = 2\pi$. We conclude, as above, that the condition for the stability of rotation of the unbalanced mass with the phase ψ_* has the form

$$\frac{dP}{d\psi_*} = -\frac{\pi}{\lambda}(1+\varepsilon)\cos\psi_* < 0 \tag{60}$$

From here it follows that rotation with the phase ψ_{*1} is stable, and that with the phase ψ_{*2} is unstable. Note that when $\lambda \to 0$, $\psi_{*1} = 0$ and $\psi_{*2} = \pi$. It seems therefore appropriate to call the motion with the phase ψ_{*1} the in-phase motion and that with the phase ψ_{*2} the antiphase motion.

Let us now consider the more general problem of ascertaining the possibility of sustaining the rotation of the unbalanced mass at a mean angular velocity which, in the general case, is different from the angular frequency of the sinusoidal excitation (39). We shall seek the solution of Eq. (42) that corresponds to the rotation of the unbalanced mass at the mean angular velocity $\nu\omega$ where ν is an arbitrary real number different from zero. With $\nu > 0$ the unbalanced mass rotates in the direction of the exciting motion of the axis describing elliptical trajectory (39); with $\nu < 0$ it rotates in the opposite sense. We put accordingly

$$\varphi = \nu\tau + \psi \tag{61}$$

where ψ is an alternating-sign function whose derivative $\dot{\psi}$ is an alternating-sign function with zero mean value. Equation (42) takes now the form

$$\ddot{\psi} + \lambda\dot{\psi} + \mu\left[\sin(\nu\tau + \psi)\cos\tau - \varepsilon\cos(\nu\tau + \psi)\sin\tau\right] + \nu\lambda = 0 \tag{62}$$

The solution of this equation will be sought in the form of series (46); substituting it into the left-hand side of Eq. (62) and

grouping the terms with equal powers of the parameter μ, we obtain the following identity (cf. Section 19):

$$[\ddot{\psi}^{(0)}+\lambda\dot{\psi}^{(0)}]+\mu[\ddot{\psi}^{(1)}+\lambda\dot{\psi}^{(1)}-\Phi^{(1)}]+\mu^2[\ddot{\psi}^{(2)}+\lambda\dot{\psi}^{(2)}-\Phi^{(2)}]+ \\ +\ldots+\nu\lambda\equiv 0 \qquad (63)$$

where

$$\Phi^{(1)}=\Phi^{(1)}[\tau,\ \psi^{(0)}(\tau)],\ \ \Phi^{(2)}=\Phi^{(2)}[\tau,\ \psi^{(0)}(\tau),\ \psi^{(1)}(\tau)]\ \ldots \qquad (64)$$

Equating sequentially to zero each of the expressions in brackets on the left-hand side of identity (63), we obtain the equations of the zero, first, second and further approximations. One must only establish in which of the equations the free term $\nu\lambda$ should be included. For this purpose we turn to the structure of expressions (62) and (63) which show that the functions $\Phi^{(1)}$, $\Phi^{(2)}$, . . ., and consequently the function ψ as well as its components $\psi^{(0)}, \psi^{(1)}$, $\psi^{(2)}$ must be periodic with rational ν and almost-periodic with irrational ν.

Therefore the term $\nu\lambda$ cannot be included in the equation of the zero approximation which has the form (48). Its unique solution satisfying the requirement that the function $\psi^{(0)}$ be periodic or almost-periodic is a constant:

$$\psi^{(0)}=\text{const} \qquad (65)$$

We now take up the equation of the first approximation assuming that the term $\nu\lambda$ can be ascribed to it:

$$\ddot{\psi}^{(1)}+\lambda\dot{\psi}^{(1)}=\Phi^{(1)}(\tau,\ \psi^{(0)})-\frac{\nu\lambda}{\mu} \qquad (66)$$

where

$$\Phi^{(1)}(\tau,\ \psi^{(0)})\equiv-\frac{1-\varepsilon}{2}\sin[(\nu+1)\tau+\psi^{(0)}]-\frac{1+\varepsilon}{2}\sin[(\nu-1)\tau+\psi^{(0)}] \qquad (67)$$

For $\psi^{(1)}$ to be a periodic or almost-periodic function the following condition must be fulfilled:

$$\frac{1}{\tau_*}\int_0^{\tau_*}\left[\Phi^{(1)}(\tau,\ \psi^{(0)})-\frac{\nu\lambda}{\mu}\right]d\tau=0 \qquad (68)$$

where τ_* is the period of the function $\Phi^{(1)}$, if the latter has any, or $\tau\to\infty$ if $\Phi^{(1)}$ is an almost-periodic function.

With $\nu\neq\pm1$ the function $\Phi^{(1)}$ has no constant component and therefore condition (68) is reduced to the requirement that $\nu\lambda$ be ascribed to an equation of the higher approximation orders. However, the investigation of the functions $\Phi^{(2)}$, $\Phi^{(3)}$, . . . shows that with $\nu\neq\pm1$ they do not contain constant components either and therefore the term $\nu\lambda$ cannot be ascribed to them.

Thus, we have established that with sinusoidal excitation and energy dissipation, i.e., at $\lambda > 0$, the vibratory sustaining of rotation at a mean angular velocity which is different in absolute value from the angular frequency of excitation is impossible. In the case when there is no energy dissipation, i.e., at $\lambda = 0$, the initial phase $\psi^{(0)}$ has no definite value for the given ν and takes a value depending on the initial conditions. Since neither the mean angular velocity $\nu\omega$ (here ν is an arbitrary quantity) nor the initial phase $\psi^{(0)}$ of the unbalanced mass depends on the vibratory excitation, there is no sustaining of rotation by vibration in this case at all.

The situation is quite different in two special cases where $\nu = \pm 1$, i.e., when the absolute values of the angular frequency of excitation and the angular velocity of rotation are equal. In these cases

$$\Phi^{(1)}(\tau, \psi^{(0)}) \equiv -\frac{1 \mp \varepsilon}{2} \sin(2\tau + \psi^{(0)}) - \frac{1 \pm \varepsilon}{2} \sin \psi^{(0)}. \tag{69}$$

The upper sign in (69) and in the expressions below corresponds to $\nu = 1$ (the sense of rotation of the unbalanced mass and that of the exciting elliptical motion of the axis are the same), and the lower sign corresponds to $\nu = -1$ (the sense of rotation of the unbalanced mass is opposite to that of the elliptical motion of the axis). At $\nu = 1$ formula (69) and the subsequent relations yield results that are identical with those presented by expressions (51) through (60).

Substituting the right-hand side of expression (69) into condition (68), we have

$$\sin \psi^{(0)} = \mp \frac{2\lambda}{\mu(1 \pm \varepsilon)} \tag{70}$$

Hence the necessary condition for the vibratory sustaining of the rotation of the unbalanced mass is

$$\lambda < \frac{\mu(1 \pm \varepsilon)}{2} \tag{71}$$

So, firstly, the quantity λ at which the vibratory sustaining of rotation is provided must have an order not higher than that of the small parameter μ; secondly, with $\nu = -1$ the conditions for the sustaining of rotation are stricter than with $\nu = 1$, and for the limiting case when $\varepsilon = 1$, i.e., with a circular path of exciting vibrations, the sustaining of rotation in the direction opposite to that of the exciting motion is not realized.

Inserting into inequality (71) the values of λ and μ from expressions (43), we obtain the restriction imposed on the system parameters:

$$K < \frac{1 \pm \varepsilon}{2} m_0 r a \omega \tag{72}$$

Equation (70) has two fundamentally different solutions:

$$\psi_1^{(0)} = \mp \sin^{-1} \frac{2\lambda}{\mu(1 \pm \varepsilon)}, \quad \psi_2^{(0)} = \pi \pm \sin^{-1} \frac{2\lambda}{\mu(1 \pm \varepsilon)} \tag{73}$$

Only the first of these solutions is realized as it satisfies the stability condition

$$\frac{d}{d\psi^{(0)}}\left\{\frac{1}{\tau_*}\int_0^{\tau_*}\left[\Phi^{(1)}(\tau, \psi^{(0)}) \mp \frac{\lambda}{\mu}\right]d\tau\right\}<0 \tag{74}$$

whence

$$\cos\psi^{(0)}<0 \tag{75}$$

Accurate to terms of the order of μ

$$\psi(\tau)=\psi^{(0)}\pm\frac{1\mp\varepsilon}{8}\sin(2\tau\pm\psi^{(0)}) \tag{76}$$

Expression (76) with $\nu = 1$ is identical with expression (56) and shows that with circular excitation, when $\varepsilon = 1$, the unbalanced mass rotates at a constant angular velocity ω.

With rectilinear exciting vibrations, i.e., when $\varepsilon = 0$, both cases, $\nu = 1$ and $\nu = -1$, become identical.

37. Optimum Shape of Unbalances

The problem of determining the most advantageous shape of unbalanced masses is met with in solving a number of dynamic problems and in designing centrifugal vibration generators. In some cases it is required to ensure the smallest possible overall dimensions and the minimum weight of a vibration machine. There are other cases when the aim is to accelerate the transient processes at starting and running-down of the vibration machine, in order, for example, to reduce increases in vibration on transition through intermediate resonances or to ensure the attainment of working conditions with a motor having but small excess power. Cases are met with when the nonuniformity of the rotation of unbalanced masses should be either increased or reduced under steady operating conditions. An increase in the nonuniformity is required, for instance, in designing superharmonic centrifugal vibration generators and its reduction may be useful in shock-and-vibration machines where the operating conditions of the motors become worse because of jumps in the angular velocity of the unbalanced masses which arise at shocks (cf. Section 42).

Similar problems arise in designing balance weights which are unbalanced masses whose rotation generates centrifugal forces used to counterbalance other undesirable inertia forces caused by the operation of a machine.

In solving concrete problems of optimization of the shape of unbalanced masses a variety of requirements are to be met and diverse additional conditions to be satisfied. But, as a rule, all

these problems are variational and isoperimetric. In order to make the problems more definite we shall consider the following three:

1. To ensure the minimum (maximum) weight of the unbalance for a given static mass moment with respect to the axis of rotation. This is equivalent to ensuring the maximum (minimum) static mass moment of the unbalanced mass for a given weight.

2. To ensure the minimum (maximum) moment of inertia of the unbalance with respect to its axis of rotation, given its static mass moment with respect to this axis. This is equivalent to ensuring the maximum (minimum) static mass moment, given the moment of inertia.

3. To ensure minimization (maximization) of the criterion α^2 defined by one of the equalities (2), Sec. 34, on which the degree of uniformity of unbalance rotation depends.

The ratios whose minimization or maximization is to be attained by the solution of the problems depend substantially on the length of the unbalance and its density. We shall assume these quantities to be known, the density of the unbalanced material to be the same at all its points and the shape of the unbalance to be prismatic. The last assumption makes it possible to replace the problem by a plane one and to consider, instead of the mass, static mass moment and mass moment of inertia of the unbalance, its area, static moment of the area and moment of inertia of the area. Therefore the criteria whose minimization or maximization is required in the three problems formulated above may be written in the following form, respectively:

$$\lambda = \frac{F+F_0}{K} \tag{1}$$

$$\mu = \frac{J+J_0}{K} \tag{2}$$

$$\frac{1}{\alpha^2} = \frac{(J+J_0)(F+F_0+F_n)}{K^2} \tag{3}$$

where F and F_0 = areas of the unbalanced and balanced parts of the unbalance, respectively

F_n = area corresponding to the mass of the parts of the vibration machine which perform rectilinear vibrations

J and J_0 = moments of inertia with respect to the axis of rotation of the areas of the unbalanced and balanced parts of the unbalance, respectively

K = static moment of the area of the unbalance with respect to the axis passing through the axis of rotation at right angles to the line which connects the axis of rotation with the centre of gravity of the unbalance area.

Whatever the criterion, the optimum shape of the unbalance (Fig. 87*a*) must be symmetric with respect to the straight line Oy passing through the axis of rotation O and the centre of gravity C of unbalance *1234561*. Therefore we shall consider only the right-hand half of the unbalance, assuming that it is entirely above the x-axis, i.e., meaning the outline of only the unbalanced part of the out-of-alignment rotor (unbalance).

We formulate now the problem as follows. Let a part of the unbalance contour, for instance, the segments ab and cd (Fig. 87*b*) be represented by the given curves. It is required to determine the shape of the missing part of the contour that would ensure the minimization or maximization of one of the criteria: λ, μ, $1/\alpha^2$. Let us treat the case of the minimization of the criterion μ.

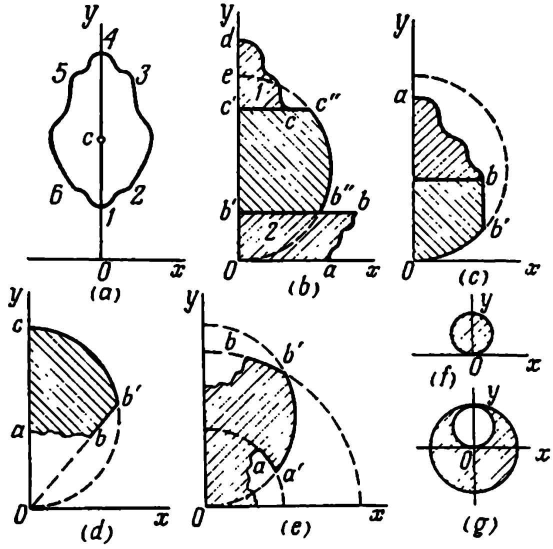


Figure 87

We draw the lines bb' and cc' parallel to the x-axis and write down the expressions of the moment of inertia and of the static moment of the unbalance:

$$\left.\begin{aligned} J &= J_1 + J_2 + \int_{y_b}^{y_c} \left(\frac{1}{3}x^3 + xy^2\right) dy \\ K &= K_1 + K_2 + \int_{y_b}^{y_c} xy\, dy \end{aligned}\right\} \tag{4}$$

where J_1 and K_1 = moment of inertia and static moment of the given figure *2* ($Oabb'$)

J_2 and K_2 = moment of inertia and static moment of the given figure *1* ($c'cd$).

The min $\left(\frac{J+J_0}{K}\right)$ is to be determined. We have therefore to study the variational isoperimetric problem which is reduced to finding the min $(J - pK)$, i.e., to solving the equation

$$\delta(J - pK) = 0 \tag{5}$$

where δ = symbol of the variational operation

p = parameter determined from an additional condition.

Making use of the Euler-Lagrange's equation, we write down the required equation of the extremal problem

$$x^2+y^2-py=0$$

or

$$x^2+(y-r)^2=r^2 \tag{6}$$

where

$$r=\frac{1}{2}p \tag{7}$$

Thus, the required part of the contour consists of the arc $b''c''$ which is part of the half-circle $Ob''c''e$ tangent to the x-axis at point O and of the straight-line segments $c'c$, $c'b'$ and $b'b''$. The parameter r can be determined, for example, by specifying the value of the static moment K. The discontinuities in the abscissas bb'' and cc'' are, generally speaking, inevitable. Moreover, the problem may have no solution for some combinations of the given portions *1* and *2* and value of K.

If the end-point abscissas of the required part of the contour are given instead of the ordinates, then the discontinuities will occur in the ordinate. For example, let the upper part ab (Fig. 87c) of the unbalance contour be given; the outline of the lower part of the contour is to be determined. In this case, in calculating the values of J and K one must integrate with respect to x:

$$\left.\begin{aligned} J &= \int_0^{x_b}\left[\frac{1}{3}(y_2^3-y_1^3)+(y_2-y_1)\,x^2\right]dx \\ K &= \int_0^{x_b}\frac{1}{2}(y_2^2-y_1^2)\,dx \end{aligned}\right\} \tag{8}$$

where $y_2(x)$ and $y_1(x)$ are the given upper and the sought-for lower parts of the contour, respectively.

In this case, in order to minimize the criterion μ it is necessary that the required part of the contour Ob' be represented by the curve (6) and the discontinuity $b'b$ be, in the general case, in the ordinate. This is readily ascertained by solving the problem by the method indicated above.

The result obtained, which states that the required section of the contour ensuring the minimization of the criterion μ is an arc forming part of the half-circle tangent to the x-axis at point O, remains valid if the problem is treated with the use of other reference systems, including the curvilinear ones. In the polar coordinate system the required part of the contour may be represented, for example, by the portion $bb'c$ (Fig. 87d) or $aa'b'b$ (Fig. 87e).

Of all the shapes of the unbalance at a given static moment K, the most advantageous in respect of the minimization of the criterion μ is a complete circle tangent to the abscissa axis (Fig. 87*f*). Indeed, substituting in the integral on the right-hand side of the first of Eqs. (4) the value of K expressed in terms of r with the aid of Eq. (6) and equating to zero the derivatives of this integral with respect to the upper and lower limits, we find the most advantageous limits of integration:

$$y_c = 2r, \quad y_b = 0 \tag{9}$$

It is clear that this result also determines the most advantageous variant of maximization of the criterion μ with the same outer radius of the unbalance as in the preceding case. The shape of the unbalance is represented by the shaded part of the area in Fig. 87*g*.

In order to establish the condition for the minimization of the criterion $1/\alpha^2$ with the statement of the problem formulated above we now write, in addition to equalities (4), the expression of the unbalance area:

$$F = F_1 + F_2 + \int_{y_b}^{y_c} x\,dy \tag{10}$$

where F_1 and F_2 are the areas of the given parts *1* and *2* in Fig. 88*a*.

It is required to find

$$\min\left[\frac{(J+J_0)(F+F_0+F_n)}{K^2}\right]$$

this being reduced to solving the equation

$$\delta(J - pK + qF) = 0 \tag{11}$$

where p and q are parameters to be determined from additional conditions.

In this case the equation of the extremal has the form

$$x^2 + y^2 - py + q = 0$$

whence, using the notations

$$l = \frac{1}{2}p, \quad r = \sqrt{\frac{1}{4}p^2 - q} \tag{12}$$

we obtain

$$x^2 + (y - l)^2 = r^2 \tag{13}$$

Thus, the required part of the contour of the unbalance consists, as in the case of the minimization of the criterion μ, of the part-arc $b''c''$ of the half-circle $fb''c''e$ subtended by the y-axis and of the segments bb'' and cc''. This case differs from the preceding one in that

the distance l of the centre of the half-circle from the origin is greater than its radius r.

Inserting the value of x from Eq. (13) into the integral on the right-hand side of the first of equalities (4) and equating to zero the derivatives of the integral with respect to its limits, we find that with given F and K the most advantageous unbalance has the shape of a full circle. Indeed, the most advantageous limits of integration prove to be

$$y_c = l + r, \quad y_b = l - r \tag{14}$$

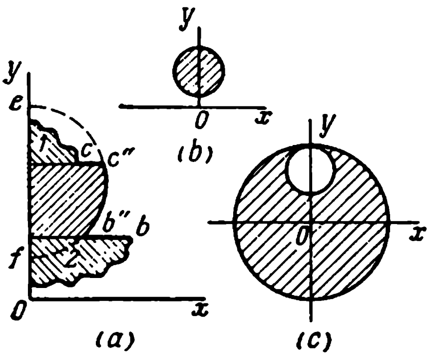


Figure 88

Figure 88*b* shows the most advantageous shape of unbalance that serves to minimize the criterion $1/\alpha^2$. The most advantageous shape for the maximization of this criterion for the same outer radius of the unbalance is given in Fig. 88*c*.

Let us briefly discuss the minimization of the criterion λ. This problem is trivial. Indeed, the static moment

$$K = F\bar{y}$$

where $\bar{y}$ is the ordinate of the centre of gravity of the area F. Therefore

$$\lambda = \frac{1}{\bar{y}}$$

and, consequently, whatever the given curve describing the outer (more remote from the x-axis) part of the unbalanced-mass contour, it is necessary, in order to minimize the criterion λ, that the part of the contour nearer to the x-axis be a straight line parallel to the abscissa axis, since this condition ensures the maximum distance of the centre of gravity of the unbalance from the abscissa axis.

38. Some Features of Frictional-Planetary Vibration Generators

In the frictional-planetary vibration generator schematically shown in Fig. 68*c* or *d*, the constraint imposed by the race in the generator body on the runner is unilateral (unretaining).

As pointed out in Section 30, at the point of contact between runner and race both the normal and tangential components of the runner reaction are transmitted to the body. Thus, for the normal operation of the frictional-planetary vibration generator to be ensured with the runner neither losing contact nor slipping in its

rolling over the race two necessary conditions must be observed: the sign of the normal reaction must not change; and the coefficient of sliding friction between runner and race must not be less than the maximum absolute value of the ratio of the tangential to the normal component of the reaction.

Frictional-planetary vibration generators with an unbalanced runner are also employed; this makes for rapid excitation of rolling operation and non-single-frequency vibrations of the working member. The schematic of such a vibration generator is given in Fig. 89. Runner *1* rolls over race *2* in body *3*. The runner is rotated by a motor not shown in the figure. The geometric centre of the runner A does not coincide with its centre of gravity G.

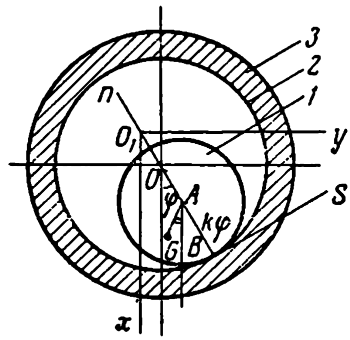


Figure 89

The runner out-of-balance with respect to the axis of its proper rotation causes oscillations of the normal and tangential components of the reaction and may lead to a noticeable nonuniformity of the rotation of the runner. In this connection it would be of interest to investigate the motion of such a system. We assume the system to be centred, i.e., that the mass centre of the working member (the generator body) coincides with the centre O of the race and the resultant reaction of the medium passes through the point O. We assume also that the working member is immersed in a viscous isotropic medium, the rolling of the runner over the race does not involve energy dissipation and that the vibrator body is capable of only translational motion.

The system under consideration has three degrees of freedom provided that the contact between runner and race is continuously maintained and no slipping occurs. We place the origin of cartesian coordinates xO_1y at the point O_1 which is the mean point about which the centre O of the working member oscillates.

The kinetic energy of the system is given by

$$T=\frac{1}{2}m_1\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]+\frac{1}{2}J_0k^2\left(\frac{d\varphi}{dt}\right)^2+$$
$$+\frac{1}{2}m_0\left\{\left[\frac{dx}{dt}-\frac{d\varphi}{dt}(r\sin\varphi+p\sin k\varphi)\right]^2+\right.$$
$$\left.+\left[\frac{dy}{dt}+\frac{d\varphi}{dt}(r\cos\varphi-p\cos k\varphi)\right]^2\right\} \tag{1}$$

where x, y = coordinates of the centre of gravity of working member (i.e., of the point O)

$r = OA$ = modulus of the radius-vector of the geometric centre of runner relative to the race centre

$p = AG$ = modulus of the radius-vector of the centre of gravity of runner with respect to its geometric centre

φ = turning angle of the radius-vector r measured counterclockwise from the positive direction of the x-axis

$k\varphi$ = turning angle of the radius-vector p measured clockwise from the positive direction of the x-axis

k = ratio of the angular velocity modulus of the rolling motion of runner over the race to the modulus of the angular velocity of the proper rotational motion of runner, i.e., the reciprocal of the transmission ratio i defined by formula (19), Sec. 30

m_1 and m_0 = masses of working member and runner, respectively

J_0 = central moment of inertia of runner.

The motor characteristic is expressed by formula (28), Sec. 35:

$$M = M' - k_1 \frac{d\varphi}{dt} + k_2 \frac{d^2\varphi}{dt^2}$$

If we select as dimensionless variables

$$\tau = \omega t; \quad \xi = \frac{m_1 + m_0}{m_0 r} x; \quad \zeta = \frac{m_1 + m_0}{m_0 r} y \tag{2}$$

where $\omega = (d\varphi/dt)_{mean}$ is the mean value of the angular velocity of the rolling motion, and introduce the dimensionless parameters

$$\left.\begin{aligned} &\varepsilon = \frac{p}{r}, \quad \beta = \frac{b}{2(m_1+m_0)\omega} \\ &\alpha^2 = \frac{(m_0 r)^2}{(m_1+m_0)[J_0 + m_0(r^2+p^2) - k_2]} \\ &\lambda = \frac{m_0 r p}{J_0 + m_0(r^2+p^2) - k_2}, \quad \mu_0 = \frac{M'}{[J_0 + m_0(r^2+p^2) - k_2]\omega^2} \\ &\eta = \frac{k_1}{J_0 + m_0(r^2+p^2) - k_2} \end{aligned}\right\} \tag{3}$$

where b is the coefficient of dissipative resistance to the vibration of the generator body, then the differential equations of motion of the system take the form

$$\left.\begin{aligned} &\ddot{\xi} + 2\beta\dot{\xi} - \ddot{\varphi}(\sin\varphi + \varepsilon \sin k\varphi) - \dot{\varphi}^2(\cos\varphi + \varepsilon k \cos k\varphi) = 0 \\ &\ddot{\zeta} + 2\beta\dot{\zeta} + \ddot{\varphi}(\cos\varphi - \varepsilon\cos k\varphi) - \dot{\varphi}^2(\sin\varphi - \varepsilon k \sin k\varphi) = 0 \\ &\ddot{\varphi}[1 - 2\lambda\cos(1+k)\varphi] + \eta\dot{\varphi} + (1+k)\lambda\dot{\varphi}^2 \sin(1+k)\varphi - \\ &-\alpha^2\ddot{\xi}(\sin\varphi + \varepsilon\sin k\varphi) + \alpha^2\ddot{\zeta}(\cos\varphi - \varepsilon\cos k\varphi) = \mu_0 \end{aligned}\right\} \tag{4}$$

Here the dots over the variables denote differentiation with respect to τ.

We now introduce a new variable

$$\psi = \varphi - \tau \tag{5}$$

Assuming the quantity ψ and its derivatives to be small as compared to unity, we rewrite Eqs. (4), retaining only the terms of an order not higher than the first in ψ and its derivatives:

$$\left.\begin{aligned}
&\ddot{\xi}+2\beta\dot{\xi}=\cos\tau+\varepsilon k\cos k\tau+(\ddot{\psi}-\psi)\sin\tau+\\
&+\varepsilon(\ddot{\psi}-k^2\psi)\sin k\tau+2\dot{\psi}\cos\tau+2\varepsilon k\dot{\psi}\cos k\tau\\
&\ddot{\zeta}+2\beta\dot{\zeta}=\sin\tau-\varepsilon k\sin k\tau-(\ddot{\psi}-\psi)\cos\tau+\\
&+\varepsilon(\ddot{\psi}-k^2\psi)\cos k\tau+2\dot{\psi}\sin\tau-2\varepsilon k\dot{\psi}\sin k\tau\\
&\ddot{\psi}+\eta\dot{\psi}=\mu_0-\eta-(1+k)\lambda\sin(1+k)\tau+\lambda[2\ddot{\psi}-\\
&-(1+k)^2\psi]\cos(1+k)\tau-2(1+k)\lambda\dot{\psi}\sin(1+k)\tau+\\
&+\alpha^2\ddot{\xi}(\sin\tau+\varepsilon\sin k\tau)-\alpha^2\ddot{\zeta}(\cos\tau-\varepsilon\cos k\tau)
\end{aligned}\right\}\qquad(6)$$

We shall take as a zero approximation to ψ the periodic solution of the third (abridged) of Eqs. (6):

$$\ddot{\psi}+\eta\dot{\psi}=-(1+k)\lambda\sin(1+k)\tau\qquad(7)$$

which takes the following form (the zero approximation is denoted by an asterisk):

$$\psi^*=-\psi_a^*\sin[(1+k)\tau-\theta^*]\qquad(8)$$

where

$$\psi_a^*=\frac{\lambda}{\sqrt{(1+k)^2+\eta^2}};\quad\theta^*=-\tan^{-1}\frac{\eta}{1+k}\qquad(9)$$

Inserting solution (8) into the first two equations (6) and integrating them, we obtain the first approximations:

$$\left.\begin{aligned}
&\xi=a_1\cos(\tau+\tau_1)+a_2\cos(k\tau+\tau_2)-\\
&-a_3\cos[(1+2k)\tau-\tau_3]-a_4\cos[(2+k)\tau-\tau_4]\\
&\zeta=a_1\sin(\tau+\tau_1)-a_2\sin(k\tau+\tau_2)+\\
&+a_3\sin[(1+2k)\tau-\tau_3]-a_4\sin[(2+k)\tau-\tau_4]
\end{aligned}\right\}\qquad(10)$$

where

$$\left.\begin{aligned}
&a_1=\sqrt{\frac{\left(1+\frac{1}{2}\varepsilon\psi_a^*\cos\theta^*\right)^2+\left(\frac{1}{2}\varepsilon\psi_a^*\sin\theta^*\right)^2}{1+4\beta^2}}\\
&a_2=\sqrt{\frac{\left(\varepsilon+\frac{1}{2}k\psi_a^*\cos\theta^*\right)^2+\left(\frac{1}{2}k\psi_a^*\sin\theta^*\right)^2}{k^2+4\beta^2}}\\
&a_3=\frac{\left(\frac{1}{2}+2k+2k^2\right)\varepsilon\psi_a^*}{(1+2k)\sqrt{(1+2k)^2+4\beta^2}};\quad a_4=\frac{\left(2+2k+\frac{1}{2}k^2\right)\psi_a^*}{(2+k)\sqrt{(2+k)^2+4\beta^2}}
\end{aligned}\right.$$

$$\left.\begin{aligned}
\tau_1 &= \tan^{-1}\frac{\frac{1}{2}\varepsilon\psi_a^*\sin\theta^*}{1+\frac{1}{2}\varepsilon\psi_a^*\cos\theta^*}+\tan^{-1}2\beta\\
\tau_2 &= \tan^{-1}\frac{\frac{1}{2}k\psi_a^*\sin\theta^*}{\varepsilon+\frac{1}{2}k\psi_a^*\cos\theta^*}+\tan^{-1}\frac{2\beta}{k}\\
\tau_3 &= \theta^*-\tan^{-1}\frac{2\beta}{1+2k};\quad \tau_4=\theta^*-\tan^{-1}\frac{2\beta}{2+k}
\end{aligned}\right\}\tag{11}$$

Expressions (10) show that even the first approximation reveals four vibration frequencies of the working member.

Substituting solutions (10) and zero approximation (8) on the right-hand side of the third of Eqs. (6), we obtain a linear differential equation with constant coefficients. To ensure the periodicity of its solution we equate to zero the constant terms on the right-hand side of the equation. The result is the power balance equation:

$$\mu_0-\eta-\alpha^2a_1\sin\tau_1-\frac{1}{2}(1+k)^2\lambda\psi_a^*\sin\theta^*=0 \tag{12}$$

Solving this equation (upon changing to dimensional parameters), we find the mean angular velocity ω of the rolling motion of the runner.

Upon integrating the differential equation obtained from the third of Eqs. (6) we find the first approximation to ψ. We do not give it here because the derivation is rather cumbersome due to the presence of a great number of sinusoidal components of dimensionless frequencies k, $1-k$, 1, $1+k$, $1+2k$, $1+3k$, $2k$, 2, $2+k$, $2+2k$, $3+k$, $3+2k$.

The runner takes the body reaction at the point B of contact between runner and race; this reaction is equal to the product of the mass of the runner by the absolute acceleration of its centre of gravity. Denoting the projections of the reaction on the x- and y-axes by the corresponding capital letters, X and Y, we can write their expressions as follows:

$$\left.\begin{aligned}
X &= m_0\,[\ddot{x}-\ddot{\varphi}\,(r\sin\varphi+p\sin k\varphi)-\dot{\varphi}^2\,(r\cos\varphi+kp\cos k\varphi)]\\
Y &= m_0\,[\ddot{y}+\ddot{\varphi}\,(r\cos\varphi+p\cos k\varphi)-\dot{\varphi}^2\,(r\sin\varphi-kp\sin k\varphi)]
\end{aligned}\right\}\tag{13}$$

We shall now use a new cartesian system of coordinates nBs with the origin at the point of contact B. The n-axis is directed along the normal towards the geometric centres of the runner and the race; the s-axis is directed in the sense of the rolling motion of the runner. Projecting X and Y onto the axes, we obtain the normal

N and the tangential S components of the reaction transmitted by the working member to the runner at the point of contact:

$$\left.\begin{aligned} N &= -X\cos\varphi - Y\sin\varphi \\ S &= -X\sin\varphi + Y\cos\varphi \end{aligned}\right\} \qquad (14)$$

Substituting into (14) the first approximations to x, y, φ, we can check whether the condition of maintaining of the runner contact with the race

$$N > 0 \qquad (15)$$

is satisfied at all times as well as the condition of absence of runner slipping

$$|S| < fN \qquad (16)$$

where f is the coefficient of sliding friction.

CHAPTER 7

THE DYNAMICS OF THE SHOCK-AND-VIBRATION DRIVE

39. The Simplest Shock-and-Vibration System

The vibrations of machines and instruments are often accompanied by shocks arising from the presence of clearances and unilateral constraints between the elements; these shocks lead to premature failure of machines and instruments. On the other hand, in many machines the shock-and-vibration conditions are the basis of the working process.

Shock-and-vibration systems are essentially nonlinear[1]. The study of such systems is complicated by their motion being discontinuous. However, some of the dynamic problems of shock-and-vibration systems can be solved by approximation methods that have been developed for the treatment of continuous nonlinear systems. In certain cases the differential equations of motion for shock-and-vibration systems can be effectively integrated by applying the method of fitting. The most detailed results are furnished by investigating the stability of the solutions of the differential equations, i.e., of the stability of motion under given conditions. A very general method of such studies suitable in particular for the treatment of discontinuous systems is the point mapping method which will be discussed in Section 40.

Viewing shocks as momentary phenomena they may be considered to be of several kinds. In shocks of the first kind the first derivative of the displacement with respect to time (velocity) has discontinuities with a limited range (discontinuity of the first kind); in shocks of the second kind the second derivative of the displacement with respect to time (acceleration) has a discontinuity of the first kind, etc. Often, in the narrow sense of the word, when speaking of shocks, it is the shock of the first kind that is meant.

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[1] The existence of shocks to which the system is subjected does not in itself make the system nonlinear. If a linear system is subjected to external shocks whose impulses and moments of striking are independent of the state of the system, such a system remains linear. The nonlinearity arises if the magnitudes of shock impulses and moments of striking or even one of the two factors depend on the state of the system, i.e., on its state coordinates.

Among the earlier investigations of mechanical shock-and-vibration systems the works published by I. Rusakov and A. Kharkevich should be mentioned, in which the method of fitting was used to investigate the vibrations of a system subjected to shocks of the first kind, and the work of Yu. Yorish containing the treatment of subharmonic resonances in a system with shocks of the third kind.

In the present chapter we shall be concerned with the problems of the dynamics of systems performing forced vibrations accompanied by shocks of the first kind. A preliminary consideration of the free vibrations of such a simple system (Fig. 90) seems to be expedient here. Body *1* of mass m is suspended from linear spring *2* of stiffness c and because of ideal constraints *3* can have only a motion of translation along the x-axis. Body *1* can strike while moving against fixed stop *4*.

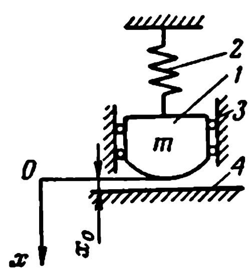

Figure 90

We shall measure the displacements of body *1* from the equilibrium position. The distance covered by body *1* from the equilibrium position with no stop to the contact with the stop will be denoted by x_0. This distance can be positive (there is a clearance between the stop and the position of equilibrium) or negative (there is an interference, i.e., the equilibrium position with no stop would be beyond the stop position). In the latter case the initial reference point corresponds to the equilibrium position which body *1* would occupy if there were no stop.

If no energy dissipation occurs during the interval between shocks, the motion is described by the differential equation

$$\ddot{x}+\omega_0^2 x=0 \tag{1}$$

where

$$\omega_0=\sqrt{\frac{c}{m}} \tag{2}$$

is the angular frequency of the system with no stop.

We take the origin of time $t=0$ to be the moment when body *1* separates from the stop. At this moment

$$x_+=x_0, \quad \dot{x}_+=-v_1 \tag{3}$$

The subscript "+" denotes the quantities x and $\dot{x}$ immediately after the shock; the subscript "—" is used to denote the same quantities just before the shock.

With the initial conditions (3) the solution of Eq. (1) may be written in the form

$$x=A_1\cos(\omega_0 t+\chi_1) \tag{4}$$

where

$$A_1=\sqrt{x_0^2+\frac{v_1^2}{\omega_0^2}}; \quad \tan\chi_1=-\frac{v_1}{x_0\omega_0} \tag{5}$$

Upon differentiating equality (4), we obtain

$$\dot{x}=-A_1\omega_0\sin(\omega_0 t+\chi_1) \tag{6}$$

At the moment $t=t_1$ when body *1* again comes into contact with the stop the state of the system is determined by the conditions

$$x_-=x_0, \quad \dot{x}_-=v_1 \tag{7}$$

Substituting conditions (7) into equalities (4) and (6) and dividing the latter by the former, we can write

$$\tan(\omega_0 t_1+\chi_1)=-\frac{v_1}{x_0\omega_0} \tag{8}$$

Comparison of expression (8) with the second of equalities (5) yields the relation

$$\omega_0 t_1+\chi_1=2\pi-\chi_1$$

whence

$$t_1=\frac{2(\pi-\chi_1)}{\omega_0}=\frac{2}{\omega_0}\left(\pi-\cos^{-1}\frac{x_0}{A_1}\right) \tag{9}$$

The initial conditions of the next (second) cycle at $t=t_1$ will be

$$x_+=x_0, \quad \dot{x}_+=-Rv_1 \tag{10}$$

where R is Newton's coefficient of velocity (or momentum) restitution characterizing the jump in velocity in a shock and defined by the relation

$$R=-\frac{\dot{x}_+}{\dot{x}_-}, \quad (0\leqslant R\leqslant 1) \tag{11}$$

For the second cycle we have

$$A_2=\sqrt{x_0^2+\frac{R^2v_1^2}{\omega_0^2}}; \quad t_2=\frac{2}{\omega_0}\left(\pi-\cos^{-1}\frac{x_0}{A_2}\right) \tag{12}$$

where t_2 is the duration of the second cycle.

Similarly for the nth cycle[1]

$$A_n=\sqrt{x_0^2+\frac{R^{2n-2}v_1^2}{\omega_0^2}}, \quad t_n=\frac{2}{\omega_0}\left(\pi-\cos^{-1}\frac{x_0}{A_n}\right) \tag{13}$$

[1] We assume $R=\text{const}$.

The duration of the cycle t_n can be expressed in terms of the vibration swing D_n:

$$t_n = \frac{2}{\omega_0}\left[\pi - \sec^{-1}\left(\frac{D_n}{x_0} - 1\right)\right] \tag{14}$$

where

$$D_n = A_n + x_0 \tag{15}$$

An analysis of the relations obtained shows that at $R < 1$

$$\lim_{n\to\infty} D_n = \begin{cases} 2x_0 & \text{at} \quad x_0 \geqslant 0 \\ 0 & \text{at} \quad x_0 \leqslant 0 \end{cases} \tag{16}$$

$$\lim_{n\to\infty} t_n = \begin{cases} \dfrac{2\pi}{\omega_0} & \text{at} \quad x_0 > 0 \\ 0 & \text{at} \quad x_0 < 0 \end{cases} \tag{17}$$

and at $x_0 = 0$

$$t_n = \frac{\pi}{\omega_0} = \text{const} \tag{18}$$

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} R^{n-1} v_1 = 0 \tag{19}$$

It follows that with $x_0 > 0$ in the system treated, with decreasing vibration swing, the duration of the cycles increases, which is characteristic of systems with a hardening curve of the restoring force (cf. Section 21). On the contrary, at $x_0 < 0$ with decreasing swing the duration of the cycle becomes shorter, which is characteristic of systems with a softening curve of the restoring force. With $x_0 = 0$ the system becomes isochronous.

If $R = 1$, the system becomes a conservative one and expression (9) yields the vibration period. With $R = 0$ the vibration swing and the duration of the cycle (in the given case, the duration of the period) take the limiting values given by formulas (16), (17) after the first shock.

Introducing the "dimensionless duration" τ_n of the cycle and the "dimensionless clearance" δ_n for free vibrations defined by the expressions

$$\tau_n = \omega_0 t_n, \quad \delta_n = \frac{x_0}{A_n} \tag{20}$$

we have from equality (9)

$$\tau_n = 2\left(\pi - \cos^{-1}\delta_n\right) \tag{21}$$

The "dimensionless clearance" may vary within the limits $-1 \leqslant \delta \leqslant 1$; with the lower limit the vibrations stop, with the upper they become shockless. Figure 91 shows τ_n as a function of δ_n.

A shock-and-vibration system even in the simplest case of sinusoidal excitation may have many qualitatively different motion regimes. Their study, especially the ascertainment of the conditions under which the stability of a certain regime is ensured, calls for the application of sufficiently powerful methods. Such an investigation making use of the point mapping method is carried out in Section 41.

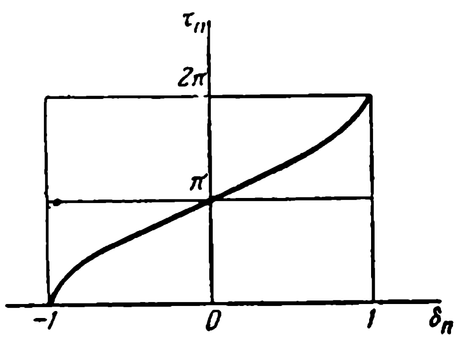

Figure 91

In a shock-and-vibration system performing periodic forced vibrations the period of the vibrations which is always a multiple of (in a special case, is equal to) the exciting force period will be more or less close to the duration of the cycle of the free vibrations of the system with a swing equal to that of forced vibrations. This is a consequence of the entrainment effect which is strongly pronounced in shock-and-vibration systems.

40. Point Mapping Method

It has been shown in the preceding chapters that the investigation of linear vibratory systems fits completely into the general scheme which has been elaborated in detail. The great variety of nonlinear vibratory systems precludes the possibility of working out an analogous detailed scheme for their investigation. Nevertheless there is a general and very fruitful approach to these systems based on some fundamental theorems of the qualitative theory of differential equations — the so-called point mapping method. In conjunction with several analytical and numerical integration methods this method permits one to study the characteristic features of the nonlinear system under consideration and gives, which is very important, a general picture of its behaviour.

The basic idea of the point mapping method is due to H. Poincaré. This method was later developed in works by A. Andronov and his collaborators and is successfully used in solving complex theoretical problems in nonlinear vibrations and automatic control. The point mapping method has been expounded in strict order and well grounded, including its application to multidimensional systems, by Yu. Neimark.

We shall demonstrate the application of the method to the study of an autonomous single-degree-of-freedom system and give some general results applicable to systems having many degrees of freedom.

We shall now discuss an autonomous systems with one degree of freedom, making for the present no special assumptions concerning its concrete properties.

It has been shown in Section 16 that only one state trajectory passes through an ordinary point on the state plane of this system. Let us draw in the state plane an arbitrary segment L which has the property that the state trajectories of the system intersect the line without touching it (Fig. 92). Let us denote the coordinate of a point of the segment L by s, measuring the coordinate from the initial point A. Let the state trajectory C at $t = t_0$ intersect the segment L at point M whose coordinate is s. If at $t > t_0$ the trajectory C does not intersect the segment L again, we say that the point M has no successor on the segment. If at a certain $\bar{t} > t_0$ the trajectory C again intersects the segment L at point $\bar{M}$ whose coordinate is $\bar{s}$, then this point is called the successor of M. It can be proved that if the point M has a successor, then all the points of L sufficiently near to M also have this property.

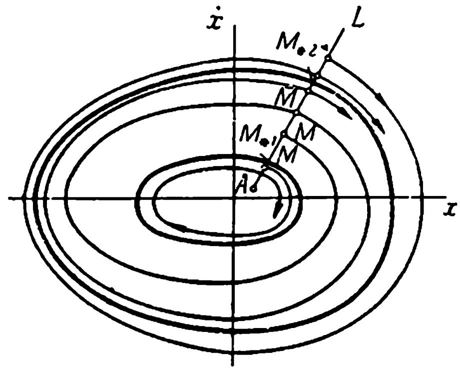

Figure 92

Thus each state trajectory C establishes a certain correspondence between the points $\bar{M}$ and M of the segment L. It may be said that the state trajectories of the system in question generate the point mapping of the segment L into itself. The symbolic representation of this mapping is

$$\bar{M} = TM \tag{1}$$

It is clear that the form of the mapping T depends on the nature of the state trajectories, i.e., on the concrete properties of the system.

Formula (1) expresses a one-to-one correspondence. This statement follows from the fact that only one state trajectory passes through each non-singular point in the state plane.

A closed state trajectory, the limit cycle, corresponds to the periodic motion of the system. For such a trajectory the points M and $\bar{M}$ coincide. A point that is transformed into itself by the mapping T is called the *fixed point* of the mapping. Such points satisfy the symbolic equation

$$M_* = TM_* \tag{2}$$

Expression (1) in its analytic form expresses the relation between the coordinates of the points M and $\overline{M}$

$$\bar{s}=f(s) \tag{3}$$

The coordinates of the fixed point corresponding to a closed state trajectory satisfy the equation

$$s_*=f(s_*) \tag{4}$$

The function $f(s)$ is called the *successor function*. Thus if the successor function is known, the problem of finding the limit cycles is reduced to finding the fixed points of the mapping, in other words, to solving Eq. (4).

The successor function makes it possible also to answer the question about the stability of the limit cycle found. All the unclosed trajectories passing in the neighbourhood of the limit cycle have the form of spirals that unwind from or wind on the limit cycle at $t\to\infty$ (Fig. 92). The limit cycle is unstable in the former case and stable in the latter.

A certain sequence of the points within the segment L corresponds to each open state trajectory. If any such sequence converges to the fixed point M_*, this fixed point is stable; otherwise it is unstable. Thus there is a one-to-one correspondence between the stability of the limit cycle and the stability of the fixed point.

Let the points M and $\overline{M}$ be near the fixed point M_*, i.e., the differences $s-s_*$ and $\bar{s}-s_*$ have small absolute values. Let us expand the right-hand side of Eq. (3) in a series in powers of the difference $s-s_*$. The resulting expression is

$$\bar{s}=f(s_*)+(s-s_*)\left(\frac{df}{ds}\right)_{s=s_*}+[\text{terms of the order of } (s-s_*)^2] \tag{5}$$

Let us denote $s-s_*$ by Δs and $\bar{s}-s_*$ by $\overline{\Delta s}$. These quantities are the distances from the point M and its successor to the fixed point M_*. If Δs is sufficiently small, the quadratic terms may be neglected. Making use of Eq. (4) one can write:

$$\overline{\Delta s}=\left(\frac{df}{ds}\right)_{s=s_*}\Delta s \tag{6}$$

Expression (6) is called the *linearized successor function*. The term is self-explanatory.

If $\left|\left(\frac{df}{ds}\right)_{s=s_*}\right|<1$, then the distance between $\overline{M}$ and M_* is less than that between M and M_*. Constructing the successor for the point $\overline{M}$ and extending the process we find that the sequence of the points

$$M;\ \overline{M}=TM;\ \overline{\overline{M}}=T\overline{M}=T^2M;\dots \tag{7}$$

approaches M_* steadily. Thus the fixed point proves stable. On the contrary, with $\left|\left(\frac{df}{ds}\right)_{s=s_*}\right| > 1$ the successors of the point M recede from the fixed point M_* and in this case it is unstable.

These results obtained by not very strict reasoning constitute the content of Koenigs' theorem. They lend themselves to a very simple and comprehensive geometric interpretation by means of the so-called Koenigs-Lameray stairway (Fig. 93)—the graphical representation of the function $\bar{s} = f(s)$. It is readily seen that the points of intersection of this graph with the straight line $\bar{s} = s$ are the fixed points of the mapping.

The construction (shown in Fig. 93) of the sequence (7) allows one to see that any such sequence beginning on the right or on the

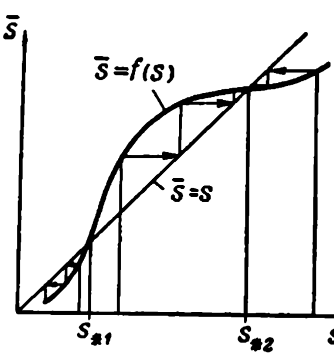

Figure 93

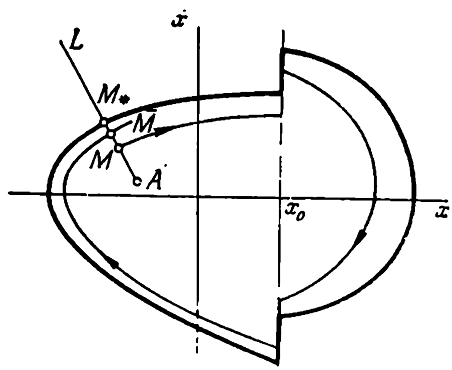

Figure 94

left of the fixed point M_{*2} (but not on the left of point M_{*1}) converges to the point M_{*2}. Thus the fixed point M_{*2} for which the condition $\left|\left(\frac{df}{ds}\right)_{s=s_*}\right| < 1$ is satisfied proves stable while the point M_{*1} is unstable. An example of the state pattern of such a system is illustrated in Fig. 92.

The above discussion is equally applicable to any dynamic systems, including shock-and-vibration systems. Let the system be described by different differential equations in different regions of the state plane; let also the transition from one region to another be accompanied by a jumpwise change in state. The state trajectories of such a system are illustrated in Fig. 94. The descriptions of the system with $x > x_0$ and $x < x_0$ are different; at $x = x_0$ the velocity $\dot{x}$ changes by a jump. It can be easily seen from geometric considerations that the successor function is continuous even in this case. This fact considerably facilitates the study.

Note that the actual construction of the successor function requires that the equation of motion be integrated by some method. In this

respect the point mapping method can yield no new results. However it serves as the conceptual basis and effective means in studying the general pattern of the motion of a dynamic system which would hardly be feasible in another way.

In applied problems point mapping is often given in parametric form, i.e., takes the form

$$s=\varphi(u),\ \bar{s}=\psi(u) \tag{8}$$

By eliminating the parameter u we obtain again the mapping in the form (3). Actually there is no need in such an elimination. In fact, all the results concerning the specific features of mapping can also be obtained in the parametric form. The value of the parameter u corresponding to the fixed point is determined by the equation

$$\varphi(u_*)=\psi(u_*) \tag{9}$$

Noting that the derivative $\frac{df}{ds}$ may be written as $\frac{d\bar{s}}{ds}$ in accordance with expression (3) and differentiating formula (8), we obtain the condition of the stability of the fixed point in the form

$$\left|\frac{d\psi}{du}\right|_* < \left|\frac{d\varphi}{du}\right|_* \tag{10}$$

The asterisk signifies that the derivatives are taken at the fixed point. Instead of the Koenigs-Lameray stairway we finally arrive at the construction indicated in Fig. 95.

Consider, as an example, the shock-and-vibration system pictured in Fig. 96. A body of mass m is suspended from a spring of stiffness c. At the moment when the system passes through the equilibrium position from left to right, the body experiences a blow which causes a momentary change in velocity. Suppose that at every impact the body acquires a constant increase in kinetic energy. This system is an extremely simplified model of the clock mechanism. Assuming the dissipative resistance to be proportional to the velocity, we may write the equation of motion for the interval between the impacts as follows:

$$m\ddot{x}+b\dot{x}+cx=0 \tag{11}$$

If $\dot{x}_-$ is the velocity just before the impact, $\dot{x}_+$ the velocity just after the impact and K the increase in kinetic energy, then

$$\text{at } x=0,\ \dot{x}>0,\ \frac{m\dot{x}_+^2}{2}-\frac{m\dot{x}_-^2}{2}=K \tag{12}$$

Since the time t enters explicitly neither into Eq. (11) nor into the additional condition (12), the system under consideration is autonomous. We take as the segment L that part of the positive semi-axis $O\dot{x}$ which begins at the origin. In this case s is represented by the velocity $\dot{x}$ at $x=0$. Consider the state trajectory issuing

from the point M $(0, \dot{x}_0)$ of the axis. Upon integrating Eq. (11) we obtain the equation of the state trajectory in parametric form for the interval between shocks:

$$\left.\begin{aligned} x &= \frac{\dot{x}_0}{\omega} e^{-ht} \sin \omega t \\ \dot{x} &= \dot{x}_0 e^{-ht}\left(\cos \omega t - \frac{h}{\omega} \sin \omega t\right) \end{aligned}\right\} \qquad (13)$$

where

$$h = \frac{b}{2m}; \qquad \omega = \sqrt{\omega_0^2 - h^2}; \qquad \omega_0^2 = \frac{c}{m}$$

We take as the origin of time the moment just following the shock.

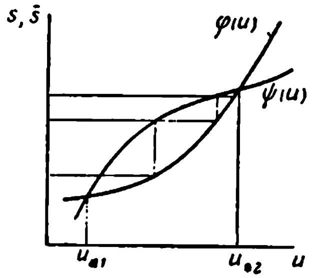

Figure 95

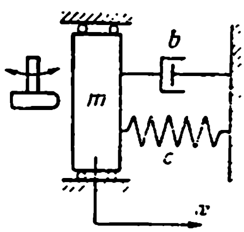

Figure 96

At $t = 2\pi/\omega$ the representative point passes from the left half-plane to the right one. The next blow is dealt at this moment. Before the shock the value of the velocity is

$$\dot{x}_- = \dot{x}_0 e^{-\vartheta} \qquad (14)$$

where the symbol ϑ is used for the logarithmic decrement of the vibrations: $\vartheta = 2\pi h/\omega$. Making use of condition (12), we find the coordinate $\dot{x}_1$ of the successor for point M:

$$\dot{x}_1 = \sqrt{\dot{x}_0^2 e^{-2\vartheta} + \frac{2}{m} K} \qquad (15)$$

Obviously the successor function (15) is continuous. Equating $\dot{x}_1$ and $\dot{x}_0$, we obtain the coordinate of the unique fixed point:

$$\dot{x}_* = \sqrt{\frac{\frac{2}{m} K}{1 - e^{-2\vartheta}}} \qquad (16)$$

The derivative $\frac{d\dot{x}_1}{d\dot{x}_0}$ at the fixed point has the value

$$\left(\frac{d\dot{x}_1}{d\dot{x}_0}\right)_* = \left(\frac{\dot{x}_0 e^{-2\vartheta}}{\sqrt{\dot{x}_0^2 e^{-2\vartheta} + \frac{2}{m} K}}\right)_* = e^{-2\vartheta} < 1 \qquad (17)$$

Consequently the system has the unique stable limit cycle that is shown in Fig. 97*b*. The Koenigs-Lameray stairway for this problem is illustrated in Fig. 97*a*.

It should be pointed out that the model discussed does not reflect a number of essential features of the clock mechanism. As can be seen from Fig. 97*a* the state trajectory with any initial velocity, including the arbitrarily small one, approaches the limit cycle with time.

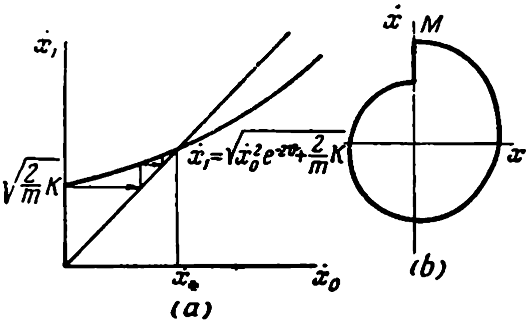

Figure 97

In other words, the system has the property of soft self-excitation. Real clock mechanisms have no such property: to set the mechanism in motion one must impart to it a certain initial impulse different from zero. A model with dry friction is more plausible.

We now turn to the exposition of some results regarding multi-degree-of-freedom systems. These results are equally applicable to autonomous and nonautonomous systems with the only difference that in the former we consider the state space and in the latter the so-called extended state space. It differs from the state space by the presence of an additional coordinate—the time. It is characteristic of nonautonomous vibrating systems that their motion is a periodic function of time with a certain period T. It is therefore expedient to introduce this additional coordinate in such a way that the moments of time that differ by multiples of T would coincide. In the case of a nonautonomous vibrating system having one degree of freedom we have to consider the cylindrical extended state space. By analogy the extended state space of a nonautonomous vibrating system having many degrees of freedom is also called *cylindrical* though in this case naturally a graphical representation is impossible. In the extended state space, as in the state space of an autonomous vibrating system, a closed limit cycle corresponds to the periodic motion.

In the following discussion, for the sake of brevity, we shall use only the term state space.

Let the motion of the representative point in the state space be described by the following set of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n, t), \qquad (i = 1, 2, \ldots, n) \tag{18}$$

where $x_1, x_2, \ldots, x_n$ are the state coordinates.

The only assumption made concerning the functions X_i is that they ensure the uniqueness of the state trajectory which passes through any point of the state space and also that they are periodic in t with the period T. We place a certain surface S in the state space and denote by $x_{10}, x_{20}, \ldots, x_{n0}$ the coordinates of a point M on this surface. Let the trajectory issuing from M again intersect the surface S at point $\overline{M}$ whose coordinates are $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$. We shall say, as we have done earlier, that the state trajectories of the set (18) realize the point mapping of the surface S into itself, and write the mapping in the form (1). The relationship between the coordinates M and $\overline{M}$ is expressed by

$$\overline{x}_i = f_i(x_{10}, x_{20}, \ldots, x_{n0}), \qquad (i = 1, 2, \ldots, n) \tag{19}$$

the form of the functions f_i being determined by differential equations (18). For the set of functions f_i, $(i = 1, 2, \ldots, n)$ we retain the earlier designation—successor functions.

If the state trajectory is a limit cycle, then the point M denoted in this case by M_* coincides with its successor and is called the *fixed point of the mapping*. Its coordinates are determined from the equations

$$x_{i*} = f_i(x_{1*}, x_{2*}, \ldots, x_{n*}), \qquad (i = 1, 2, \ldots, n) \tag{20}$$

The investigation of the stability of the fixed point is carried out similarly to the above, by linearizing the mapping in the neighbourhood of the fixed point. The generalization of Koenigs' theorem in this case is formulated as follows.

The fixed point is stable if the moduli of all the roots of the characteristic equation

$$\begin{vmatrix} \left(\frac{\partial f_1}{\partial x_{10}}\right)_* - z & \left(\frac{\partial f_1}{\partial x_{20}}\right)_* & \cdots & \left(\frac{\partial f_1}{\partial x_{n0}}\right)_* \\ \left(\frac{\partial f_2}{\partial x_{10}}\right)_* & \left(\frac{\partial f_2}{\partial x_{20}}\right)_* - z & \cdots & \left(\frac{\partial f_2}{\partial x_{n0}}\right)_* \\ \cdots & \cdots & \cdots & \cdots \\ \left(\frac{\partial f_n}{\partial x_{10}}\right)_* & \left(\frac{\partial f_n}{\partial x_{20}}\right)_* & \cdots & \left(\frac{\partial f_n}{\partial x_{n0}}\right)_* - z \end{vmatrix} = 0 \tag{21}$$

are strictly less than unity.

The parametric form of the point mapping that is more frequently met with in concrete problems is

$$\varphi_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n; \; x_{10}, x_{20}, \ldots, x_{n0}; \quad u_1, u_2, \ldots, u_m) = 0 \qquad (22)$$
$$(i = 1, 2, \ldots, n+m)$$

Expression (22) contains the parameters u_k $(k = 1, 2, \ldots, m)$ which may not be amenable to elimination in the general form. Among equations (22) is also the equation of the surface S

$$S(x_1, x_2, \ldots, x_n) = 0 \qquad (23)$$

The coordinates of the fixed point are determined from the set of equations

$$\varphi_i(x_{1*}, x_{2*}, \ldots, x_{n*}; \quad x_{1*}, x_{2*}, \ldots, x_{n*}; \quad u_{1*}, u_{2*}, \ldots, u_{m*}) = 0$$
$$(i = 1, 2, \ldots, n+m) \qquad (24)$$

The condition of the stability of the fixed point remains the same and the characteristic equation takes the following form:

$$\begin{vmatrix}
\left(\frac{\partial\varphi_1}{\partial\bar{x}_1}\right)_* z + \left(\frac{\partial\varphi_1}{\partial x_{10}}\right)_*; & \left(\frac{\partial\varphi_1}{\partial\bar{x}_2}\right)_* z + \left(\frac{\partial\varphi_1}{\partial x_{20}}\right)_*; \ldots; \\
\left(\frac{\partial\varphi_1}{\partial\bar{x}_n}\right)_* z + \left(\frac{\partial\varphi_1}{\partial x_{n0}}\right)_*; & \left(\frac{\partial\varphi_1}{\partial u_1}\right)_*; \ldots; \left(\frac{\partial\varphi_1}{\partial u_m}\right)_* \\
\left(\frac{\partial\varphi_2}{\partial\bar{x}_1}\right)_* z + \left(\frac{\partial\varphi_2}{\partial x_{10}}\right)_*; & \left(\frac{\partial\varphi_2}{\partial\bar{x}_2}\right)_* z + \left(\frac{\partial\varphi_2}{\partial x_{20}}\right)_*; \ldots; \\
\left(\frac{\partial\varphi_2}{\partial\bar{x}_n}\right)_* z + \left(\frac{\partial\varphi_2}{\partial x_{n0}}\right)_*; & \left(\frac{\partial\varphi_2}{du_1}\right)_*; \ldots; \left(\frac{\partial\varphi_2}{\partial u_m}\right)_* \\
\cdots & \cdots \\
\left(\frac{\partial\varphi_{n+m}}{\partial\bar{x}_1}\right)_* z + \left(\frac{\partial\varphi_{n+m}}{\partial x_{10}}\right)_*; & \left(\frac{\partial\varphi_{n+m}}{\partial\bar{x}_2}\right)_* z + \left(\frac{\partial\varphi_{n+m}}{\partial x_{20}}\right)_*; \ldots; \\
\left(\frac{\partial\varphi_{n+m}}{\partial\bar{x}_n}\right)_* z + \left(\frac{\partial\varphi_{n+m}}{\partial x_{n0}}\right)_*; & \left(\frac{\partial\varphi_{n+m}}{\partial u_1}\right)_*; \ldots; \left(\frac{\partial\varphi_{n+m}}{\partial u_m}\right)_*
\end{vmatrix} = 0 \qquad (25)$$

Thus, if the point mapping has been performed, then in order to find the periodic solutions of the set (18) the fixed points must be found, i.e., we have to solve the set of transcendental equations (20) or (24). Usually it is impossible to obtain the solution in the general form, with the exception of several very simple special cases, and one has to use some numerical method.

Having obtained the solution, one sets up a characteristic equation using formulas (21) or (25) and estimates its roots. One should pay attention to the following.

Expanding the determinant (21) or (25), we obtain the characteristic equation

$$a_0 + a_1 z + \ldots + a_n z^n = 0 \tag{26}$$

Its coefficients $a_0, a_1, \ldots, a_n$ are functions of the parameters of the system. If we want to investigate the behaviour of the system within a certain range of its parameters, we must study the relations between the roots of the characteristic equation and these parameters.

Let us now introduce the so-called parameter space; we shall lay off on the coordinates of this space the values of the parameters of the system (they may be, for instance, natural frequencies, mass ratios, the coefficient of velocity restitution at impact, etc.). Suppose the fixed point of the mapping (19) or (22) has been found. This point, generally speaking, is stable in a certain domain of the parameter space. Since all the elements of the determinant (21) or (25) are real, the fixed point may go beyond this domain in three cases:

(1) when with variation of the system parameters one of the roots of the characteristic equation becomes equal to $+1$;

(2) when one of the roots is -1;

(3) when there appear two complex conjugate roots whose moduli are equal to unity.

Therefore, substituting into the characteristic equation $z = +1$, $z = -1$, and $z = \cos\varphi + i \sin\varphi$, we obtain the equations of the three surfaces that constitute the boundaries of the parameter space domain in which the fixed point of the kind considered is stable. The boundaries are denoted by N_{+1}, N_{-1} and N_φ, respectively.

The surface N_{+1}, $(z = +1)$

$$a_0 + a_1 + \ldots + a_n = 0 \tag{27}$$

The surface N_{-1}, $(z = -1)$

$$a_0 - a_1 + \ldots + (-1)^n a_n = 0 \tag{28}$$

The surface N_φ, $(z = \cos\varphi + i \sin\varphi)$

$$\left.\begin{aligned} a_0 + a_1 \cos\varphi + \ldots + a_n \cos n\varphi = 0 \\ a_1 \sin\varphi + a_2 \sin 2\varphi + \ldots + a_n \sin n\varphi = 0 \end{aligned}\right\} \tag{29}$$

It can be shown that the boundary N_{+1} is at the same time the boundary of the existence of this fixed point. In other words, with $z = +1$ the set of Eqs. (20) or (24) becomes unsolvable. This is why the domain bounded by the surfaces N_{+1}, N_{-1} and N_φ is called the *domain of the existence and stability of the fixed point.*

Note that the motion of the system may be subjected to certain natural limitations which lead to the appearance of additional boundary surfaces in the parameter space. An example of such limitations will be treated in the next section.

41. Problems of the Dynamics of Vibrohammers and Vibrotampers

Figure 98 pictures the dynamic model of a nonautonomous shock-and-vibration system which has one degree of freedom. This model is an idealization of spring vibrohammers and some other shock-and-vibration machines. A sinusoidal exciting force $F = F_a \cos \omega t$ and a constant force P which may, in a particular case, be identical with the weight mg are applied to the mass m supported by ideal springs of stiffness c. The vibration of the mass m is accompanied by shocks against a fixed undeformable stop.

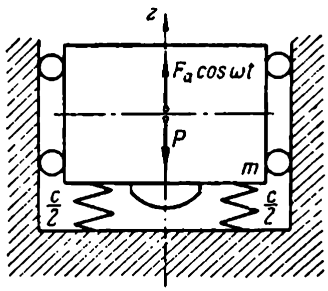

Figure 98

Various problems of the dynamics of this system have been treated by many authors. The most comprehensive investigation is due to L. Bespalova. On the basis of the preceding section we shall study some periodic motions of this system.

Let us denote the displacement of the mass m by x. Generally, in measuring x the position of static equilibrium is selected as the point of reference, which slightly simplifies the formulas. In this case we shall measure the displacement from the position in which the spring is undeformed. This will later make it possible to directly obtain results pertaining to the springless shock-and-vibration systems without solving the problem anew.

Assuming that energy dissipation occurs only with the shock, we may write the equation of motion for the interval between shocks in the following form:

$$m \frac{d^2x}{dt^2} + cx = F_a \cos \omega t - P \tag{1}$$

We assume that the shock is momentary and characterize it by the coefficient of velocity restitution R defined by relation (11), Sec. 39. Let the stop be located at the elevation $x = x_0$. At the moment of shock, i.e., at $x = x_0$, the velocity dx/dt of the mass m changes by a jump

$$\left(\frac{dx}{dt}\right)_+ = -R \left(\frac{dx}{dt}\right)_- \tag{2}$$

It is clear that the system can be in motion only above the stop. Therefore the solution that corresponds to real motion must satisfy the condition

$$x > x_0 \tag{3}$$

This is a limitation of the type mentioned at the end of Section 40.

Introducing the dimensionless variables

$$\zeta=\frac{m\omega^2}{F_a}x, \quad \tau=\omega t \tag{4}$$

we can write Eq. (1) and condition (2) in the following form:

$$\ddot{\zeta}+\gamma^2\zeta=\cos\tau-p \tag{5}$$

$$\dot{\zeta}_+=-R\dot{\zeta}_- \quad \text{at} \quad \zeta=\zeta_0 \tag{6}$$

Thus the system can be characterized by the following four dimensionless parameters:

$$\gamma=\sqrt{\frac{c}{m\omega^2}}; \quad p=\frac{P}{F_a}; \quad R; \quad \zeta_0=\frac{m\omega^2}{F_a}x_0 \tag{7}$$

Consider the motion of the representative point in a three-dimensional cylindrical state space of our system (Fig. 99). Let us introduce in this space the cylindrical coordinates

$$r=\zeta+r_0; \quad z=\dot{\zeta}; \quad \varphi=\tau \tag{8}$$

where r_0 is an arbitrary constant which satisfies the condition $\zeta_0+r_0>0$ within the range of variation of ζ_0.

Each integral curve in this space consists of segments of two types. The first type is represented by spiral-like curves issuing from some point on the upper half of the cylinder $r=\zeta_0+r_0$, ($\dot{\zeta}>0$) and arriving at a certain point on its lower half ($\dot{\zeta}<0$). These segments satisfy Eq. (5). To the other type of trajectory parts belong segments of the generatrix of the cylinder rising up from below; the vertical coordinates of the initial and end points of each segment are related by condition (6).

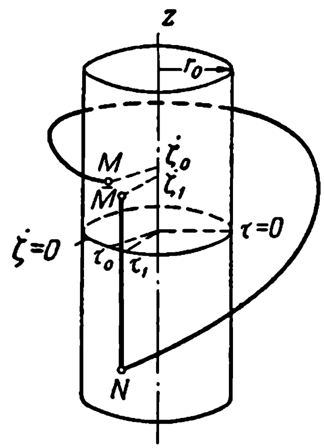

Figure 99

The closed integral curves consisting of alternating segments of the two types correspond to the periodic motions. It is readily seen that the number of the segments of the first or second type is simply the number of shocks within one period of motion; we denote this number by s. Shock-and-vibration systems have a property that is characteristic of other nonlinear systems as well: the period of motion may be not only equal to but be a multiple of the period of the exciting force. With the notations used in (4) the period of the exciting force is 2π, and the period of motion $2\pi n$, ($n=1, 2, \ldots$). At $n>1$ the motion is called subharmonic of the order of $1/n$.

The periodic motion is characterized by two integers, s and n. We shall use the notation D_{sn} for the domain of parameter values which correspond to periodic motions with identical s, n. Motions with one shock per period, i.e., those belonging to the domain D_{1n} are of the greatest interest in practice.

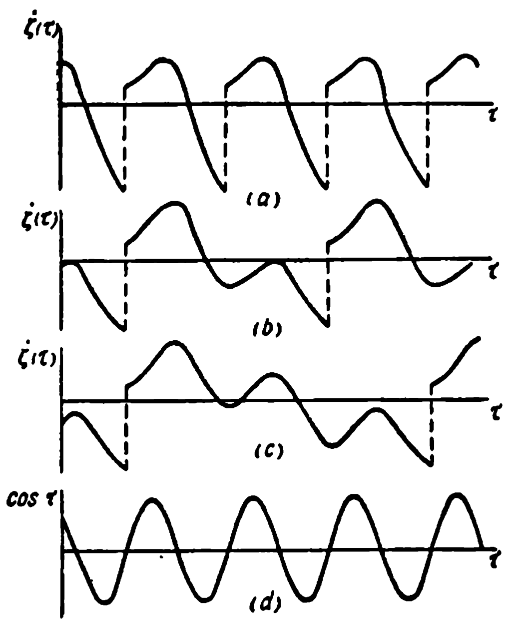

Figure 100

The time-histories of the velocity $\dot{\zeta}$ shown in Fig. 100 correspond to diverse periodic motions with one shock per period: curve a to the domain D_{11}, curve b to D_{12}, and curve c to D_{13}. The curve d in this figure represents the exciting force $\cos\tau$.

Let us perform the point mapping of the cylinder L into itself by means of state trajectories approaching the periodic motion with one shock per period $2\pi n$. Let M be a point on the upper half of the cylinder and the coordinates of the point be ζ_0, $\dot{\zeta}_0$, τ_0. The integral curve issuing from the point is described by the parametric equations

$$\left.\begin{aligned} \zeta(\tau) &= -\frac{p}{\gamma^2}+\left(\zeta_0+\frac{p}{\gamma^2}+\frac{\cos\tau_0}{1-\gamma^2}\right)\cos\gamma(\tau-\tau_0)+ \\ &\quad+\frac{1}{\gamma}\left(\dot{\zeta}_0-\frac{\sin\tau_0}{1-\gamma^2}\right)\sin\gamma(\tau-\tau_0)-\frac{1}{1-\gamma^2}\cos\tau \\ \dot{\zeta}(\tau) &= -\gamma\left(\zeta_0+\frac{p}{\gamma^2}+\frac{\cos\tau_0}{1-\gamma^2}\right)\sin\gamma(\tau-\tau_0)+ \\ &\quad+\left(\dot{\zeta}_0-\frac{\sin\tau_0}{1-\gamma^2}\right)\cos\gamma(\tau-\tau_0)+\frac{1}{1-\gamma^2}\sin\tau \end{aligned}\right\} \tag{9}$$

furnished by the integration of Eq. (5).

Let the integral curve intersect the cylinder L at the point N at $\tau=\tau_1>\tau_0$. The values of τ_1 is determined from the first of Eqs. (9) upon substituting into it $\zeta(\tau_1)=\zeta_0$; having obtained τ_1, we can find the value of $\dot{\zeta}$ at this moment from the second of Eqs. (9); let us denote this value by $\dot{\zeta}_-$. Condition (6) transfers the point N to the point $\overline{M}$ whose coordinates are

$$\zeta_0;\quad \tau_1;\quad \dot{\zeta}_1=-R\dot{\zeta}_- \tag{10}$$

Performing the operations indicated above, we represent the point mapping which transfers M into $\overline{M}$ in the following form:

$$\left.\begin{aligned}
&\varphi_1(\dot{\zeta}_1\ \tau_1;\ \dot{\zeta}_0,\ \tau_0) \equiv -\left(\zeta_0+\frac{p}{\gamma^2}\right)+\\
&\qquad+\left(\zeta_0+\frac{p}{\gamma^2}+\frac{\cos\tau_0}{1-\gamma^2}\right)\cos\gamma(\tau_1-\tau_0)+\\
&\qquad+\frac{1}{\gamma}\left(\dot{\zeta}_0-\frac{\sin\tau_0}{1-\gamma^2}\right)\sin\gamma(\tau_1-\tau_0)-\frac{1}{1-\gamma^2}\cos\tau_1 \quad 0\\
&\varphi_2(\dot{\zeta}_1,\ \tau_1;\ \dot{\zeta}_0,\ \tau_0) \equiv \frac{\dot{\zeta}_1}{R}-\gamma\left(\zeta_0+\frac{p}{\gamma^2}+\frac{\cos\tau_0}{1-\gamma^2}\right)\times\\
&\times\sin\gamma(\tau_1-\tau_0)+\left(\dot{\zeta}_0-\frac{\sin\tau_0}{1-\gamma^2}\right)\cos\gamma(\tau_1-\tau_0)+\frac{1}{1-\gamma^2}\sin\tau_1=0
\end{aligned}\right\} \tag{11}$$

The successor function is given here in implicit form corresponding to formulas (22), Sec. 40. The parameters u_i in the case being considered are absent. Had we considered motions with s shocks per period, we should have obtained $2s$ equations in coordinates of the $s-1$ intermediate points on the cylinder L, these coordinates being the parameters u_i.

Let us now find the coordinates of the fixed point of the mapping (11); for this purpose we substitute into equalities (11) $\dot{\zeta}_1=\dot{\zeta}_0=\dot{\zeta}_*$, $\tau_0=\tau_*$, $\tau_1=\tau_*+2\pi n$. From the equations obtained we find first of all expressions for $\sin\tau_*$ and $\cos\tau_*$:

$$\left.\begin{aligned}
\frac{\sin\tau_*}{1-\gamma^2}&=-\frac{\dot{\zeta}_*(1-R)}{2R}\\
\frac{\cos\tau_*}{1-\gamma^2}&=-\zeta_0-\frac{p}{\gamma^2}+\frac{\dot{\zeta}_*(1-R)}{2R}f
\end{aligned}\right\} \tag{12}$$

where

$$f=\frac{1+R}{1-R}\cdot\frac{\operatorname{cotan}\pi n\gamma}{\gamma}$$

We introduce ξ as a temporary notation for $\dot{\zeta}_*(1-R)/R$. Squaring the relations (12) and adding them up, we obtain the following quadratic:

$$(1+f^2)\xi^2-2f\left(\zeta_0+\frac{p}{\gamma^2}\right)\xi+\left(\zeta_0-\frac{p}{\gamma^2}\right)^2-\frac{1}{(1-\gamma^2)^2}=0 \tag{13}$$

It will be shown in the following that of the two solutions of Eq. (13) that with the plus sign before the root is to be taken. Consequently the dimensionless velocity just after the shock in periodic motion is expressed in terms of the system parameters as follows:

$$\dot{\zeta}_*=\frac{2R}{1-R}\cdot\frac{f\left(\zeta_0+\frac{p}{\gamma^2}\right)+\sqrt{\frac{1+f^2}{(1-\gamma^2)^2}-\left(\zeta_0+\frac{p}{\gamma^2}\right)^2}}{1+f^2} \tag{14}$$

Replacing in the first of Eqs. (9) the state coordinates of the point M by the coordinates of the fixed point M_* furnished by expressions (12) and (14), we can write the equation which describes the periodic motion in the interval between shocks as follows:

$$\zeta(\tau) = -\frac{p}{\gamma^2} + A\cos\gamma(\tau-\tau_*-\pi n) - \frac{1}{1-\gamma^2}\cos\tau \tag{15}$$

where

$$A = \frac{\dot{\zeta}_*(1+R)}{2R\gamma\sin\pi n\gamma}$$

Note that usually instead of $\dot{\zeta}_*$ the formulas contain the quantity Ru equal to it, where u denotes the absolute value of the velocity just before the shock. In this case all the formulas remain valid also at $R=0$ since u is not zero, though $\dot{\zeta}_*$ naturally vanishes.

The next step is to check the stability of the formal solution (15). In this case the characteristic equation (25) takes the form

$$\begin{vmatrix} \left(\frac{\partial\varphi_1}{\partial\dot{\zeta}_1}\right)_* z + \left(\frac{\partial\varphi_1}{\partial\dot{\zeta}_0}\right)_* & \left(\frac{\partial\varphi_1}{\partial\tau_1}\right)_* z + \left(\frac{\partial\varphi_1}{\partial\tau_0}\right)_* \\ \left(\frac{\partial\varphi_2}{\partial\dot{\zeta}_1}\right)_* z + \left(\frac{\partial\varphi_2}{\partial\dot{\zeta}_0}\right)_* & \left(\frac{\partial\varphi_2}{\partial\tau_1}\right)_* z + \left(\frac{\partial\varphi_2}{\partial\tau_0}\right)_* \end{vmatrix} = 0 \tag{16}$$

In order to calculate the elements of this determinant let us find the values of the partial derivatives of functions (11) at the fixed point.

$$\left(\frac{\partial\varphi_1}{\partial\dot{\zeta}_1}\right)_* = 0;\quad \left(\frac{\partial\varphi_1}{\partial\dot{\zeta}_0}\right)_* = \left[\frac{1}{\gamma}\sin\gamma(\tau_1-\tau_0)\right]_* = \frac{\sin 2\pi n\gamma}{\gamma}$$

$$\left(\frac{\partial\varphi_1}{\partial\tau_1}\right)_* = \left[-\gamma\left(\zeta_0+\frac{p}{\gamma^2}+\frac{\cos\tau_0}{1-\gamma^2}\right)\sin\gamma(\tau_1-\tau_0) + \right.$$

$$\left. + \left(\dot{\zeta}_0 - \frac{\sin\tau_0}{1-\gamma^2}\right)\cos\gamma(\tau_1-\tau_0) + \frac{1}{1-\gamma^2}\sin\tau_1\right]_* = -\frac{\dot{\zeta}_*}{R}$$

$$\left(\frac{\partial\varphi_1}{\partial\tau_0}\right)_* = \left[\gamma\left(\zeta_0+\frac{p}{\gamma^2}+\frac{\cos\tau_0}{1-\gamma^2}\right)\sin\gamma(\tau_1-\tau_0) - \right.$$

$$-\left(\dot{\zeta}_0 - \frac{\sin\tau_0}{1-\gamma^2}\right)\cos\gamma(\tau_1-\tau_0) - \frac{\sin\tau_0}{1-\gamma^2}\cos\gamma(\tau_1-\tau_0) -$$

$$\left. -\frac{\cos\tau_0}{1-\gamma^2}\cdot\frac{\sin\gamma(\tau_1-\tau_0)}{\gamma}\right]_* = \frac{\dot{\zeta}_*(1+R)}{2R} +$$

$$+\frac{\dot{\zeta}_*(1-R)}{2R}\cos 2\pi n\gamma - \frac{\cos\tau_*}{1-\gamma^2}\cdot\frac{\sin 2\pi n\gamma}{\gamma}$$

$$\left(\frac{\partial\varphi_2}{\partial\dot{\zeta}_1}\right)_* = \frac{1}{R};\quad \left(\frac{\partial\varphi_2}{\partial\dot{\zeta}_0}\right)_* = [\cos\gamma(\tau_1-\tau_0)]_* = \cos 2\pi n\gamma$$

$$\left.\begin{aligned}
\left(\frac{\partial\varphi_2}{\partial\tau_1}\right)_* &= \Big[-\gamma^2\left(\zeta_0+\frac{p}{\gamma^2}+\frac{\cos\tau_0}{1-\gamma^2}\right)\cos\gamma(\tau_1-\tau_0)-\\
&\quad-\gamma\left(\dot\zeta_0-\frac{\sin\tau_0}{1-\gamma^2}\right)\sin\gamma(\tau_1-\tau_0)+\frac{\cos\tau_1}{1-\gamma^2}\Big]_*=\\
&= -\frac{\gamma^2\dot\zeta_*(1-R)f}{2R}+\frac{\cos\tau_*}{1-\gamma^2}\\
\left(\frac{\partial\varphi_2}{\partial\tau_0}\right)_* &= \Big[\gamma^2\left(\zeta_0+\frac{p}{\gamma^2}+\frac{\cos\tau_0}{1-\gamma^2}\right)\cos\gamma(\tau_1-\tau_0)+\\
&\quad+\gamma\left(\dot\zeta_0-\frac{\sin\tau_0}{1-\gamma^2}\right)\sin\gamma(\tau_1-\tau_0)+\gamma\frac{\sin\tau_0}{1-\gamma^2}\sin\gamma(\tau_1-\tau_0)-\\
&\quad-\frac{\cos\tau_0}{1-\gamma^2}\cos\gamma(\tau_1-\tau_0)\Big]_*=\\
&=\frac{\dot\zeta_*(1-R)\gamma^2}{2R}\left(f-\frac{\sin 2\pi n\gamma}{\gamma}\right)-\frac{\cos\tau_*}{1-\gamma^2}\cos 2\pi n\gamma
\end{aligned}\right\}\tag{17}$$

Substituting the expressions obtained in the determinant (16) and expanding it, we get the following characteristic equation:

$$uz^2+\Big\{u\Big[(1-R)^2\sin^2\pi n\gamma+(1+R)^2\frac{\cos^2\pi n\gamma}{\gamma^2}-(1+R^2)\Big]-\\
-\left(\zeta_0+\frac{p}{\gamma^2}\right)(1+R)\frac{\sin 2\pi n\gamma}{\gamma}\Big\}z+uR^2=0 \tag{18}$$

Solution (15) is stable if the moduli of both roots of the quadratic equation (18) are less than unity. This condition can be easily expressed in terms of the coefficients of the quadratic

$$az^2+bz+c=0$$

whose roots are

$$z_1=-\frac{b}{2a}+\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}};\quad z_2=-\frac{b}{2a}-\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}$$

taking $a>0$ without loss in generality. We now seek the conditions under which $|z_1|<1$, $|z_2|<1$. Three cases are to be distinguished:

(1) $b<0$, $b^2>4ac$. In this case $|z_1|>|z_2|$ and since $z_1>0$ the required condition is reduced to

$$-\frac{b}{2a}+\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}<1$$

Hence we obtain

$$a+b+c>0 \tag{19}$$

(2) $b>0$, $b^2>4ac$. Here we have $|z_2|>|z_1|$, $z_2<0$. Similarly to the above we find the required condition to be

$$a-b+c>0 \tag{20}$$

(3) $b^2 < 4ac$. In this case the moduli of both roots are equal. They are less than unity if

$$a - c > 0 \tag{21}$$

Replacing the sign of inequality in conditions (19) through (21) by the sign of equality, we obtain of course the equations (27), (28) and (29), Sec. 40, for the boundaries N_{+1}, N_{-1} and N_φ. However the conditions found here directly contain some additional information.

Returning to Eq. (18) and inserting its coefficients into condition (19), we obtain

$$u\left[(1-R)^2 \sin^2 \pi n\gamma + (1+R)^2 \frac{\cos^2 \pi n\gamma}{\gamma^2}\right] - \left(\zeta_0 + \frac{p}{\gamma^2}\right) \frac{\sin 2\pi n\gamma}{\gamma} > 0 \tag{22}$$

The expression on the left-hand side of inequality (22) is the square root in expression (14) which can be readily verified by direct substitution. Thus we ascertain that the plus sign before the square root in expression (14) has been chosen correctly.

It was pointed out in Section 40 that the boundary N_{+1} is at the same time the boundary of the existence of a solution. In the case under consideration the condition of the existence of solution (15) is that the expression (14) for $\dot{\zeta}_*$ must be real. Therefore the equation of the existence boundary takes the form

$$\left(\zeta_0 + \frac{p}{\gamma^2}\right)^2 = \frac{1+f^2}{(1-\gamma^2)^2} \tag{23}$$

In accordance with the above statement condition (22) yields an identical result if the sign of equality is inserted into it.

The equation of the boundary N_{-1} obtained from condition (20) may be written as follows:

$$u\left[(1-R)^2 \sin^2 \pi n\gamma + (1+R)^2 \frac{\cos^2 \pi n\gamma}{\gamma^2} - 2(1+R^2)\right] = \\ = \left(\zeta_0 + \frac{p}{\gamma^2}\right)(1+R)\frac{\sin 2\pi n\gamma}{\gamma} \tag{24}$$

Finally, condition (21) yields

$$1 - R^2 > 0 \tag{25}$$

It follows that in this case the boundary N_φ, $R = 1$ coincides with the limitation deducible from the physical meaning of the coefficient of velocity restitution.

No account has so far been taken of condition (3) which, with the new notations, takes the form $\zeta > \zeta_0$. This condition is violated if in the interval between shocks the mass m touches the stop. The possibility of this occurrence is illustrated by Fig. 101 which shows

graphs of displacements within one period for different points in the domain D_{11}. The set of parameters corresponding to curve *3* is the boundary set since in this case condition (3) is violated. Thus, this condition defines one more boundary in the parameter space. It is sometimes called the *boundary of definition* of the solution of the type being considered and denoted by C_τ. It can be readily seen from Fig. 101 that C_τ is determined by solving jointly the equations

$$\zeta(\beta)=\zeta_0; \quad \dot{\zeta}(\beta)=0 \tag{26}$$

which define the moment $\tau=\beta+\tau_*$ when condition (3) is violated[1]. Since, in general, the elimination of β proves unfeasible, Eqs. (26) should be considered as the parametric equations of the boundary C_τ. In the problem being treated they take the following form:

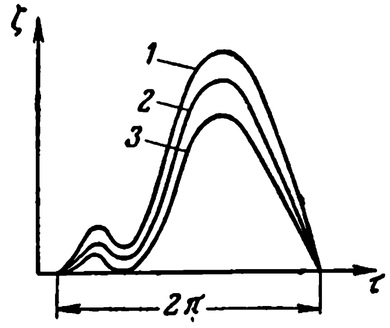

Figure 101

$$\left.\begin{aligned} A\cos\gamma(\beta-\pi n)-\frac{1}{1-\gamma^2}\cos(\beta+\tau_*)&=\zeta_0+\frac{p}{\gamma^2}\\ -\gamma A\sin\gamma(\beta-\pi n)+\frac{1}{1-\gamma^2}\sin(\beta+\tau_*)&=0\end{aligned}\right\} \tag{27}$$

In practice calculations by these formulas prove very time-consuming. They are considerably simplified in the frequently encountered case when $\zeta_0+p/\gamma^2=0$. After simple transformations Eqs. (27) become

$$\left.\begin{aligned} \frac{1-R}{1+R}&=\frac{1}{\cos\beta}\left[\cos\gamma\beta+\frac{\operatorname{cotan}\pi n\gamma}{\gamma}(\sin\beta-\gamma\sin\gamma\beta)\right]\\ \frac{1-R}{1+R}&=\frac{1}{\sin\beta}\left[\sin\gamma\beta+\frac{\operatorname{cotan}\pi n\gamma}{\gamma}(\cos\gamma\beta-\cos\beta)\right]\end{aligned}\right\} \tag{28}$$

As can be seen from expressions (7), the parameter space of the system is four-dimensional. Therefore, in order to study the space structure we have to consider several crosscuts through it. Note that the parameters ζ_0 and p enter into all the formulas only in the combined form ζ_0+p/γ^2 (in taking the position of static equilibrium as the origin it is this quantity that is usually called the clearance). Therefore we shall consider first of all the crosscut through the parameter space by the surface $\zeta_0+p/\gamma^2=0$.

The domains of existence and stability of periodic motions with one shock per period and with the periods 2π, 4π and 6π (i.e., D_{11},

[1] The necessary and sufficient condition for the existence of the boundary C_τ is that some roots should be within the interval $0<\beta<2\pi n$.

D_{12} and D_{13}) thus obtained in the plane of parameters γ, R are shown in Fig. 102. All the three domains are of one type. With $R = 1$ they are bounded by the common boundary N_φ. The boundary on the right of each domain consists of two parts. At $R = 0$ the curves C_τ issue from the points $\gamma = 1/2n$ and with a certain value of R intersect a branch of the boundary N_{-1}. The two parts of the boundary are particularly conspicuous in the domain D_{11}. The boundary on the left of each domain is formed by the second branch of the curve N_{-1}.

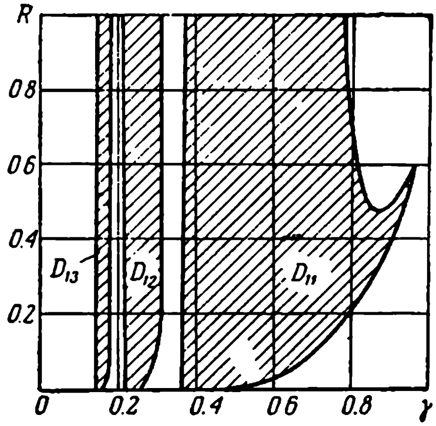

Figure 102

The boundary N_{+1} is simply non-existent in this case as can be seen from its equation (23). It should be noted that with an increase in n the dimensions of the domains are very much reduced; this is characteristic of nonlinear systems which have subharmonic solutions.

The dependence of the regime of motion on the clearance value is of practical importance since in real machines the tuning for the required regime is performed within certain limits just by

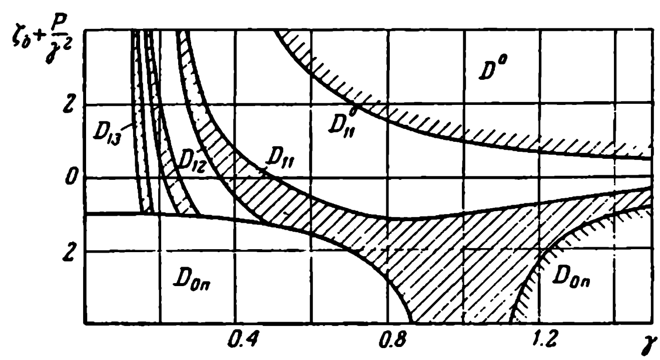

Figure 103

changing the clearance. Figure 103 shows the intersection of the same domains D_{11}, D_{12}, D_{13} with the plane of parameters γ, $\zeta_0 + p/\gamma^2$ at $R = 0$.

The following domains are marked in the figure by shading along the boundary line:

(1) the domain D_0 in which $\gamma^2 (\zeta_0 + p/\gamma^2) > 1$. In this domain the force exerted by the springs in pressing the mass to the stop

exceeds the amplitude value of the exciting force and there is no motion at all;

(2) the domain D_{0n} in which the amplitude of periodic vibration without shocks, equal to $\frac{1}{|1-\gamma^2|}$, is less than $\zeta_0 + \frac{p}{\gamma^2}$. In this domain, motions with or without shocks are possible depending on the initial conditions.

Between the domains D_0 and D_{11} there is the domain D_{11}^0 at $R = 0$ in which the motion is performed with pauses. In fact, the velocity $\dot{\zeta}_*$ just after the shock is zero. Let us denote the phase of the exciting force at this moment by τ_*^0. If it turns out that $\cos \tau_*^0 < \gamma^2 (\zeta_0 + p/\gamma^2)$, then the mass will stay pressed to the stop up to the moment τ_* when

$$\cos \tau_* = \gamma^2 \left(\zeta_0 + \frac{p}{\gamma^2}\right) \tag{29}$$

Each domain D_{1n} has on its right a similar adjacent domain D_{1n}^0 of motion with pauses. The rest of the space is occupied by domains of more complex motions with many shocks per period and higher multiple motions. We shall not dwell on the character of the sequence of these domains and on the determination of their boundaries, the more so as the parameters of the machine are selected within the domains D_{1n} where maximum shock velocities are attained.

The qualitative picture of the distribution of the domains in the γ, $\zeta_0 + \frac{p}{\gamma^2}$ plane at $R \neq 0$ remains the same. Since in this case there are no motions with pauses, the place of the domain D_{11}^0 is taken by the domain D_{s1} in which one strong and one or more weak shocks occur during the period 2π. The domains D_{1n}^0 undergo similar changes. One can form a limited conception of the changes in the dimensions and relative positions of the domains D_{1n} at $R \neq 0$ by comparison with Fig. 102.

In designing a shock-and-vibration machine its parameters should be selected so as to ensure maximum shock velocities. It has been stated above that in this respect the regimes with one shock per period of the D_{1n} type are the most favourable. The question arises how to select the optimum parameter values within the domain in question. The answer is readily found if one considers that the clearance is the design parameter whose variation is most easily attained without affecting the other parameters. Let us find the clearance value $(\zeta_0)_{opt}$ to which, in a motion with one shock per period where the period is $2\pi n$, there corresponds the maximum of the shock velocity $u = \dot{\xi}_*/R$. Differentiating expression (14) with respect

to ζ_0 and equating the result to zero, we find that

$$\left(\zeta_0+\frac{p^2}{\gamma^2}\right)^2=\frac{f^2}{(1-\gamma^2)^2}$$

Thus we obtain two optimum values of the clearance:

$$\zeta_0=-\frac{p}{\gamma^2}\pm\frac{f}{1-\gamma^2}$$

By calculating the second derivative one can readily demonstrate that the maximum shock velocity takes place when

$$(\zeta_0)_{opt}=-\frac{p}{\gamma^2}+\frac{f}{1-\gamma^2} \tag{30}$$

Of course it is necessary that the set of parameters satisfying the relation (30) be within the domain D_{1n}. With small stiffness values of the supporting springs, i.e., with small γ, to which, as can be seen from Fig. 103, correspond large n values, the point ensuring optimum velocity may prove to be outside the domain of existence and stability of the regime being considered.

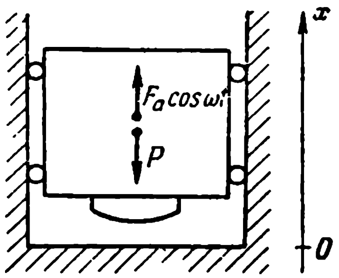

Figure 104

Finally, let us consider one more special case corresponding to $\gamma=0$. In this case we have the springless shock-and-vibration system schematically pictured in Fig. 104. This is the dynamic model of the simplest vibrotamper and springless vibrohammer. A similar scheme is used in treating some problems of vibratory conveying.

The equation of motion of the mass m in the interval between shocks is obtained from Eq. (5) by setting $\gamma=0$ in the latter:

$$\ddot{\zeta}=\cos\tau-p \tag{31}$$

The condition (6) for the shock remains unchanged. One could repeat the whole reasoning, applying it to Eq. (31): construct the point mapping, find its fixed points, etc. We shall use another approach and successively obtain all the results required by passage to the limit.

First of all it is obvious that the concept of clearance in this case is meaningless and one must put $\zeta_0=0$ in all the formulas.

Further, with small γ values we have

$$f=\frac{1+R}{1-R}\cdot\frac{\cot\text{an}\,\pi n\gamma}{\gamma}\approx\frac{1+R}{1-R}\,\frac{1}{\pi n\gamma^2\left(1+\frac{1}{3}\pi^2n^2\gamma^2\right)}$$

In the expression $1+f^2$ contained in formula (14), with small γ values, the unity can be neglected as compared to f^2. As a result, we obtain the expression

$$\dot{\zeta}_* = \frac{2R}{1-R}\left(1+\frac{1}{3}\pi^2 n^2\gamma^2\right)\pi np + \frac{2R}{1+R}\pi n\gamma^2\sqrt{1-\left(\frac{1-R}{1+R}\pi np\right)^2} \quad (32)$$

Thus, with $\gamma=0$ the velocity immediately after the shock

$$\dot{\zeta}_* = Ru = \frac{2R}{1-R}\pi np \quad (33)$$

The first of formulas (12) yields

$$\sin\tau_* = -\frac{1-R}{1+R}\pi np \quad (34)$$

In order to transform the second of formulas (12) one must substitute the expression (32) for $\dot{\zeta}_*$ retaining the second term in it. The result is

$$\cos\tau_* = \sqrt{1-\left(\frac{1-R}{1+R}\pi np\right)^2} \quad (35)$$

This result could not have been obtained directly from formula (34) because the sign before the square root would remain indeterminate.

The equation describing the periodic motion in the interval between shocks can also be derived from Eq. (15) by using the passage to the limit. Making use of expression (32) again, we may write:

$$A = \frac{\dot{\zeta}_*(1+R)}{2R\gamma\sin\pi n\gamma} \approx \frac{p}{\gamma^2} + \sqrt{1-\left(\frac{1-R}{1+R}\pi np\right)^2} + \frac{p\pi^2 n^2}{2}$$

$$\cos\gamma(\tau-\tau_*-\pi n) \approx 1 - \frac{\gamma^2(\tau-\tau_*-\pi n)^2}{2}$$

Inserting the above expressions into formula (15), we obtain

$$\zeta(\tau) = -\frac{p(\tau-\tau_*-\pi n)^2}{2} - \cos\tau + \sqrt{1-\left(\frac{1-R}{1+R}\pi\ p\right)^2} + \frac{p\pi^2 n^2}{2} \quad (36)$$

We now proceed to the transformation of the characteristic equation (18) taking into account that in accordance with expression (32)

$$u(1+R)^2\frac{\cos^2\pi n\gamma}{\gamma^2} \approx 2\pi n(1+R)\left[\frac{p}{\gamma^2} + \sqrt{1-\left(\frac{1-R}{1+R}\pi np\right)^2}\right]$$

The terms of the order $1/\gamma^2$ cancel out and we obtain the following characteristic equation

$$pz^2 + \left[(1+R)^2\sqrt{1-\left(\frac{1-R}{1+R}\pi np\right)^2} - p(1+R^2)\right]z + pR^2 = 0 \quad (37)$$

Using formulas (27) through (29) and Eq. (37), we obtain the equations of boundaries for the domains of periodic motions, with one shock per period, in the plane of the parameters p, R:

$$p=\frac{1}{\pi n}\cdot\frac{1+R}{1-R}, \text{ (boundary } N_{+1}) \tag{38}$$

$$p=\frac{(1+R)^2}{\sqrt{4(1+R^2)^2+\pi^2 n^2(1-R^2)^2}}, \text{ (boundary } N_{-1}) \tag{39}$$

$$R=1, \text{ (boundary } N_{\varphi}) \tag{40}$$

Let us also write the equations of the boundary C_τ for which the condition $\zeta > 0$ is no longer satisfied. Differentiating relation (36) and setting $\zeta(\beta)=0$, $\dot{\zeta}(\beta)=0$ where, as above, $\beta=\tau-\tau_*$ and carrying out certain transformations, we obtain the parametric equations of C_τ in the following form:

$$\left.\begin{aligned} p&=-\frac{\sin(\tau_*+\beta)}{\pi n-\beta}\\ \frac{\pi n-\beta}{2}\sin(\tau_*+\beta)&-\cos(\tau_*+\beta)+\cos\tau_*=0\end{aligned}\right\} \tag{41}$$

Figure 105 shows the domains of the existence and stability of periodic motions with one shock per period for $n=1, 2, 3$ constructed in accordance with Eqs. (38) through (41). The vertical line $p=1$ is also a boundary: at $p>1$ the force P exceeds the amplitude of the exciting force and there is no motion. The domains D_{1n} of the motions with one shock per period are bounded on the right by the curve N_{+1} and on the left by the curve N_{-1}. The boundary C_τ cuts off but an insignificant portion from the domain D_{11} near the $R=0$ axis as shown in the enlarged view of the graph in the same figure. The boundedness in accordance with $\zeta>0$ tells on the domains D_{12} and D_{13} still less.

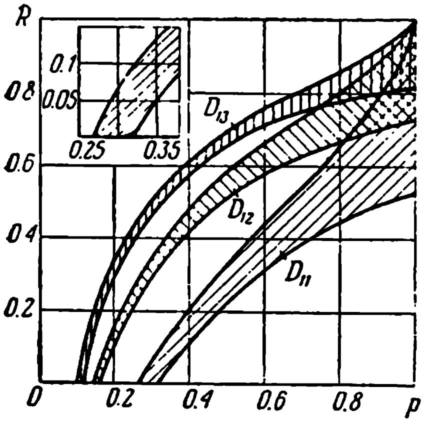

Figure 105

Note the following features of a springless shock-and-vibration system. The domains of the existence and stability of the periodic motions with one shock per period without stops of finite duration are very narrow. Their extent increases with increasing p and R;

however, it is small values of these parameters that are typical of real systems. With p and R near unity the domains overlap partially. In other words, with the same values of the parameters different periodic motions with one shock per period set in, depending on the initial conditions. Systems with small γ values have the same property at large p and R values.

The filling of the remaining part of the parameter plane has, in a general way, the same character as in the system at $\gamma \neq 0$. The domain D_{s1} extending to the right of the boundary N_{+1} of D_{11} becomes the domain D_{11}^{0} of motion with stops at $R = 0$. A similar domain is contiguous to each D_{1n} domain on the right. There are also domains of periodic motions with many shocks per period. The extent of these domains decreases as the regime becomes more complicated, i.e., with increasing s and n.

42. Some Features of Shock-and-Vibration Drives with Centrifugal Vibration Generators

Nonautonomous problems where the exciting factor is represented by a function of time, i.e., is independent of the behaviour of a vibrating system, have been treated in Sections 39 and 41. The systems excited by centrifugal vibration generators lend themselves to such idealization by postulating that the unbalanced masses rotate uniformly (cf. Section 27). In Chapter 6 we were concerned with essential problems that cannot be solved in principle if the additional degrees of freedom of the unbalanced masses are not taken into account, i.e., if the nonuniformity of their rotation is disregarded. The same features are met with in shock-and-vibration systems, sometimes in strongly accentuated form.

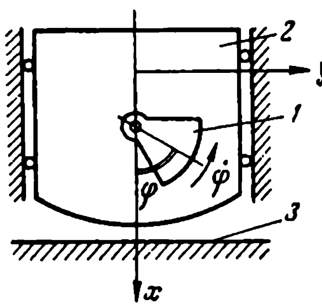

Figure 106

We start with an approximate calculation of the jump in the angular velocity of unbalance *1* (Fig. 106) whose rotation causes working member *2* to vibrate along the x-axis and strike stop *3*.

Let us denote by J_0, m_0, r the central moment of inertia, mass and eccentricity of the unbalanced mass with respect to the axis of rotation, respectively; by v the working-member velocity; by u_x, u_y the projections of the velocity of the unbalance mass centre on the axes of the coordinates; and by φ the turning angle of the radius-vector r measured from the positive direction of the x-axis; now we can write the increment of the moment of momentum and the projections L_x, L_y of the unbalance impulse during the shock

(assuming that there is no friction in the bearing) as follows:

$$\left.\begin{aligned} J_0(\dot{\varphi}_+ - \dot{\varphi}_-) &= L_x r \sin\varphi_0 - L_y r\cos\varphi_0 \\ m_0(u_{x+} - u_{x-}) &= L_x \\ m_0(u_{y+} - u_{y-}) &= L_y \end{aligned}\right\} \quad (1)$$

Here φ_0 is the angle φ at the moment of shock.

We introduce the notations

$$\left.\begin{aligned} \Delta u_x &= u_{x+} - u_{x-}; \quad \Delta u_y = u_{y+} - u_{y-} \\ \Delta v &= v_+ - v_-; \quad \Delta\dot{\varphi} = \dot{\varphi}_+ - \dot{\varphi}_- \end{aligned}\right\} \quad (2)$$

and note that in accordance with condition (10), Sec. 39,

$$\Delta v = -(1+R)v_- \quad (3)$$

where R is the coefficient of velocity restitution on impact.

Taking into account the obvious kinematic relations

$$\left.\begin{aligned} \Delta u_x &= \Delta v - r\Delta\dot{\varphi}\sin\varphi_0 \\ \Delta u_y &= r\Delta\dot{\varphi}\cos\varphi_0 \end{aligned}\right\} \quad (4)$$

we obtain the following expression for the angular-velocity jump of the unbalanced mass at the moment of shock:

$$\Delta\dot{\varphi} = \frac{m_0(1+R)v_- r\sin\varphi_0}{J_0 + m_0 r^2} \quad (5)$$

In accordance with the results of Section 41 the maximum shock velocity of the working member v_{max} for vibrations with one shock per period and without pauses of finite duration in a spring vibro-hammer with sinusoidal excitation and zero initial clearance can be expressed as follows:

$$v_{max} = \frac{2m_0 r\omega}{(m_1+m_0)\left(1-\frac{1}{4n^2}\right)(1-R)} \quad (6)$$

where m_1 = mass of working member
ω = frequency of exciting force
$1/n$ = order of subharmonic vibrations.

The maximum velocity v_{max} is attained at $\varphi_0 = \pi/2$. Substituting this expression into the right-hand side of expression (5), we obtain:

$$(\Delta\dot{\varphi})_{max} = -\frac{2(m_0 r)^2(1+R)\dot{\varphi}_{mean}}{J(m_1+m_0)\left(1-\frac{1}{4n^2}\right)(1-R)} \quad (7)$$

where $J = J_0 + m_0 r^2$ is the moment of inertia of the unbalance with respect to the axis of rotation and $\dot{\varphi}_{mean}$ is the mean velocity

of rotation of the unbalanced mass equivalent to the angular frequency ω upon sinusoidal excitation.

The expression for the relative jump of the angular velocity is accordingly

$$\frac{(\Delta\dot{\varphi})_{max}}{\dot{\varphi}_{mean}} = -\frac{2\alpha^2(1+R)}{(1-R)\left(1-\frac{1}{4n^2}\right)} \tag{8}$$

where, in accordance with formula (18), Sec. 33,

$$\alpha^2 = \frac{(m_0 r)^2}{J(m_1+m_0)} \tag{9}$$

With $n=1$ and $R=0$ we have the following simple relationship:

$$\frac{(\Delta\dot{\varphi})_{max}}{\dot{\varphi}_{mean}} = -\frac{8}{3}\alpha^2 \tag{10}$$

The relationships (7), (8) and (10) are approximate. In order to obtain more accurate results it would be necessary to integrate the set of joint differential equations of motion of the working member and the unbalanced mass. After the shock the rotation of the unbalanced mass acted upon by the motor is accelerated and, besides, the unbalance performs vibrations similar to those described in Section 33.

Another important problem requiring that the rotation of the unbalanced masses be taken into account is the self-synchronization problem. Self-synchronizing centrifugal vibration generators have been successfully used to drive shock-and-vibration machines. The theoretical determination of the self-synchronization conditions in such machines differs in some features from the cases treated in Section 36 when the working member of the machine is in a state of purely vibratory motion.

The formulation of the self-synchronization problem remains in principle the same as for systems without shocks. The equations which describe the motion of the system fall into two groups: the first group describes the motion of the carrier body, i.e., the working member of the machine, the second the rotation of the unbalanced masses. The terms that reflect the effect of the motion of the working member on the rotation of the unbalanced masses contain a small factor μ which depends on the parameters of the system and characterizes the strength of the coupling between the vibration generators and the carrier body. The problem consists in the construction of solutions of the form (18), (19), Sec. 36, and in the determination of the conditions of their existence and stability. In Section 36 we applied the Poincaré-Liapunov small-parameter method to obtain the solution of the problem. In shock-and-vibration systems the vibration velocity of the working member and the angular veloci-

ties of the unbalanced masses are discontinuous functions of time. Therefore the small-parameter method in its usual form which calls for the analyticity of the right-hand side of the differential equations proves unsuitable in this case.

Neimark has developed a generalized small-parameter method for equations whose right-hand sides are discontinuous. It is based on the point mapping method and enables the results of the self-synchronization theory to be extended to cover the systems with discontinuous characteristics, including shock-and-vibration systems. Moreover, the use of special discontinuous functions enables one to reduce the defining equations and the conditions of the stability of synchronous motions to the same form as for systems without shocks, i.e., to the formulas (18) and (25), Sec. 36, respectively.

We shall first consider Heaviside's step function defined by the relations

$$\theta(\tau)=\begin{cases}0 \text{ at } \tau<0\\ 1 \text{ at } \tau>0\end{cases} \tag{11}$$

Let $\alpha<\beta$. Clearly the difference $\theta(\tau-\alpha)-\theta(\tau-\beta)$ is a function that is unity within the interval $\alpha<\tau<\beta$ and zero outside it. Multiplication of an arbitrary function by this function "cuts off" the function outside the interval $\alpha<\tau<\beta$. If a certain function $\zeta(\tau)$ is expressed within the interval $\tau_0<\tau<\tau_1$ by $\zeta_1(\tau)$ and within the interval $\tau_1<\tau<\tau_2$ by $\zeta_2(\tau)$, etc., it can then be formally represented by the sum

$$\begin{aligned}\zeta(\tau)=[\theta(\tau-\tau_0)-\theta(\tau-\tau_1)]\,\zeta_1(\tau)+\\+[\theta(\tau-\tau_1)-\theta(\tau-\tau_2)]\,\zeta_2(\tau)+\dots\end{aligned} \tag{12}$$

and the expression (12) will hold with any τ value.

We now determine the derivative of the function $\theta(\tau)$ which we denote by $\delta(\tau)$. Since, by definition,

$$\frac{d\theta(\tau)}{d\tau}=\delta(\tau) \tag{13}$$

the relation

$$\int_{-\alpha}^{\beta}\delta(\tau)\,d\tau=\theta(\beta)-\theta(-\alpha)=1 \tag{14}$$

must be valid at any $\alpha>0$, $\beta>0$.

With $\tau<0$ and $\tau>0$ the function $\theta(\tau)$ is constant, and therefore with these τ values $\delta(\tau)$ is zero. At the point $\tau=0$ the function $\delta(\tau)$ must become infinite and in such a way that the integral (14) be equal to unity. The function $\delta(\tau)$ so defined is called *Dirac's δ-function.*

The functions $\theta(\tau)$ and $\delta(\tau)$ allow one to treat formally the discontinuous functions as continuous; they are made use of in

solving various physical and technical problems. Note that in such problems the δ-function, which has at first sight such unusual properties, appears only in the intermediate calculation steps but never enters into the final result.

A useful conception of the nature of the δ-function can be obtained by considering it as the limit of a certain sequence of the "usual" continuous functions. Let, for example,

$$f_n(\tau)=\frac{1}{\pi}\cdot\frac{n}{1+n^2\tau^2} \tag{15}$$

then $\lim\limits_{n\to\infty} f_n(\tau)=0$ at $\tau\neq 0$; $\lim\limits_{n\to\infty} f_n(0)=\lim\limits_{n\to\infty} n/\pi=\infty$; for any positive α and β we have

$$\int_{-\alpha}^{\beta} f_n(\tau)\,d\tau=\frac{1}{\pi}(\tan^{-1}n\beta+\tan^{-1}n\alpha)$$

With $n\to\infty$ the right-hand side tends to unity. Therefore, $\lim\limits_{n\to\infty} f_n(\tau)=\delta(\tau)$. Figure 107 illustrates the functions $f_n(\tau)$ for $n=2$; 5; 10. An infinite set of function sequences can be constructed, leading in the limit to the δ-function.

Note the following properties of the delta-function resulting from its definition:

$$\int_{-\alpha}^{\beta} f(\tau)\,\delta(\tau)\,d\tau=f(0) \tag{16}$$

$$\int_{-\alpha}^{\beta} f(\tau)\,\delta(\tau-\tau_1)\,d\tau=f(\tau_1),\quad -\alpha<\tau_1<\beta \tag{17}$$

$$f(\tau)\,\delta(\tau-\tau_1)=f(\tau_1)\delta(\tau-\tau_1) \tag{18}$$

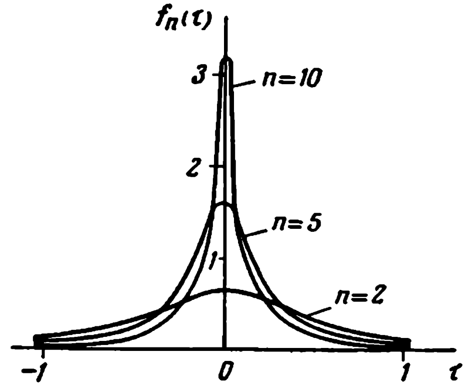

Figure 107

Here $f(\tau)$ is an arbitrary continuous function.

The proof of these formulas is straightforward. The left-hand side of relation (16) may depend only on such values of $f(\tau)$ that correspond to τ values close to zero; without making a significant error one may replace $f(\tau)$ by its value at zero, $f(0)$, which results in the basic formula (14). Formula (17) is proved by a similar reasoning and the validity of relation (18) follows directly from formula (17).

The functions $\theta(\tau)$ and $\delta(\tau)$ are very suitable for a formal description of motions accompanied by shocks. Actually, the force of the shock that is used in the classical shock theory has all the characteristic properties of the delta-function: it is different from

zero only at the moment of shock when the function becomes infinite and its integral that is equal to the impulse increment is finite.

After this digression we turn again to the self-synchronization problem. Consider a shock-and-vibration system whose body has one degree of freedom; the system is schematically pictured in Fig. 108. As distinct from the system which was treated at the beginning of Section 36 the motion of the system under consideration is accompanied by momentary shocks against a stop with the coefficient of restitution R. Since in the intervals between shocks

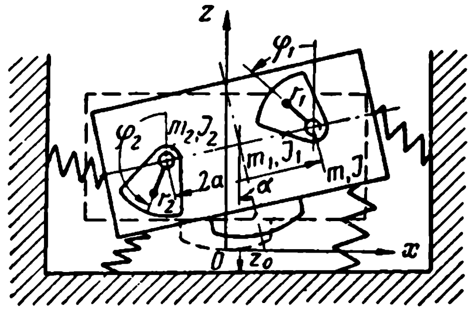

Figure 108

the system is identical with the first example in Section 36, its motion is described by the same equations which, after changing to the dimensionless variables defined by formulas (5), Sec. 36, take the following form:

$$\left.\begin{aligned} &\ddot{\zeta}+\gamma^2\zeta=[(\dot{\varphi}^2+\dot{\psi}^2)\cos\varphi+\ddot{\varphi}\sin\varphi]\cos\psi+\\ &+(\ddot{\psi}\cos\varphi-2\dot{\varphi}\dot{\psi}\sin\varphi)\sin\psi\\ &\ddot{\varphi}+\lambda\dot{\varphi}=\lambda+\mu\,[\ddot{\zeta}\sin\varphi\cos\psi+2\lambda h-\lambda h\dot{\varphi}-\\ &-\lambda(r+p)\,\dot{\psi}]\\ &\ddot{\psi}+\lambda\dot{\psi}=\mu\,[\ddot{\zeta}\cos\varphi\sin\psi+\lambda r-\lambda(r+p)\,\dot{\varphi}-\lambda h\dot{\psi}] \end{aligned}\right\} \qquad (19)$$

The condition for the shock

$$\dot{\zeta}_+=-R\dot{\zeta}_- \text{ at } \zeta=\zeta_0 \qquad (20)$$

is to be added to the equations. Since the position of static equilibrium has been taken to be the reference point from which the body displacement is measured, the dimensionless clearance ζ_0 in expression (20) coincides with the quantity ζ_0+p/γ^2 of the preceding section.

It has already been shown that subharmonic solutions are characteristic of shock-and-vibration systems. The reflection of this possible occurrence in the solution of the problem of self-synchronization

naturally suggests itself and we shall seek the solution in the form

$$\zeta=\zeta\left(\frac{\tau}{n}\right), \quad \dot{\varphi}=\tau+\gamma\left(\frac{\tau}{n}\right), \quad \psi=\psi\left(\frac{\tau}{n}\right) \tag{21}$$

where ζ, γ and ψ are functions which have the period 2π, $n=1, 2, 3, \ldots$.

The approximation of zero order in μ to the functions φ and ψ again takes the form

$$\varphi^{(0)}(\tau)=\varphi_*^{(0)}+\tau; \quad \psi^{(0)}(\tau)=\psi_*^{(0)} \tag{22}$$

It follows that the motion of the working member is described to the same approximation by the equation

$$\ddot{\zeta}+\gamma^2\zeta=\cos\psi_*^{(0)}\cos(\varphi_*^{(0)}+\tau) \tag{23}$$

and the additional condition (20). Consider the motion of the body with one shock per period $2\pi n$. If we take as the origin of time the moment of the shock, then, using the results of the preceding section, the solution of Eq. (23) is found to be

$$\zeta(\tau)=\frac{\cos\psi_*^{(0)}}{1-\gamma^2}\cos(\varphi_*^{(0)}+\tau)+A\cos\gamma(\tau-\pi n) \tag{24}$$

where

$$\left.\begin{aligned}
&\cot\text{an}\,\varphi_*^{(0)}=\frac{2\zeta_0}{u(1-R)}-f; \quad A=\frac{u(1+R)}{2\gamma\sin\pi n\gamma}\\
&f=\frac{1+R}{1-R}\cdot\frac{\cot\text{an}\,\pi n\gamma}{\gamma}\\
&u=\frac{2}{1-R}\cdot\frac{f\zeta_0+\sqrt{\dfrac{(1+f^2)\cos^2\psi_*^{(0)}}{(1-\gamma^2)^2}-\zeta_0^2}}{1+f^2}
\end{aligned}\right\} \tag{25}$$

The study of the stability of the solution obtained is carried out in the same way as it was done in Section 41. It must be stressed that this stage, the investigation of the stability of the zero approximation, is indispensable. It was omitted in Section 36 since the stability of the zero approximation in the examples considered was evident. It is further necessary to find the value of the phase $\psi_*^{(0)}$ to which corresponds the stable synchronous rotation of the unbalanced masses. It is found that for this purpose one can make use of the functions $P_1(\psi_*^{(0)})$ and $P_2(\psi_*^{(0)})$ introduced in Section 36, provided one changes suitably the form (24), in which the zero approximation must be written, so as to include the shock forces the trea tment.

Solution (24) holds only in the interval $0<\tau<2\pi n$. In the next interval of duration $2\pi n$ the first term of expression (24) remains unchanged and the second corresponding to natural vibrations

takes the form $A \cos\gamma(\tau - 3\pi n)$. Using Heaviside's function, we can give the solution the following form which holds at any $\tau > 0$:

$$\zeta(\tau) = -\frac{\cos\psi_*^{(0)}}{1-\gamma^2}\cos(\tau+\varphi_*^{(0)}) + [1-\theta(\tau-2\pi n)]\,A\cos\gamma(\tau-\pi n) + \\ + [\theta(\tau-2\pi n)-\theta(\tau-4\pi n)]\,A\cos\gamma(\tau-3\pi n) + \\ + [\theta(\tau-4\pi n)-\theta(\tau-6\pi n)]\,A\cos\gamma(\tau-5\pi n)\ldots \qquad (26)$$

In the first square brackets the number 1 replaces $\theta(\tau)$. This is because the motion is considered from the moment immediately after the shock; the initial shock is excluded from consideration. Each period consists of the interval of motion without shocks and of the shock at the end of it.

Let us calculate the dimensionless velocity $\dot{\zeta}(\tau)$. Differentiating expression (26), we obtain

$$\dot{\zeta}(\tau) = \frac{\cos\psi_*^{(0)}}{1-\gamma^2}\sin(\tau+\varphi_*^{(0)}) - [1-\theta(\tau-2\pi n)]\,A\gamma\sin\gamma(\tau-\pi n) - \\ - [\theta(\tau-2\pi n)-\theta(\tau-4\pi n)]\,A\gamma\sin(\tau-3\pi n) - \ldots + \\ + [-\delta(\tau-2\pi n)]\,A\cos\gamma(\tau-\pi n) + \\ + [\delta(\tau-2\pi n)-\delta(\tau-4\pi n)]\,A\cos\gamma(\tau-3\pi n) + \ldots \qquad (27)$$

Making use of the property (18) of the delta-function, we may rewrite the sum in (27) containing this function in the following form:

$$-\delta(\tau-2\pi n)\,A\cos\gamma(2\pi n-\pi n) + \\ + \delta(\tau-2\pi n)\,A\cos\gamma(2\pi n-3\pi n) - \delta(\tau-4\pi n)\,A\cos(4\pi n-3\pi n) + \\ + \delta(\tau-4\pi n)\,A\cos(4\pi n-5\pi n) - \ldots$$

It follows that the sum is zero at any τ. Consequently, the expression for the velocity takes the form

$$\dot{\zeta}(\tau) = \frac{\cos\psi_*^{(0)}}{1-\gamma^2}\sin(\tau+\varphi_*^{(0)}) - A\gamma\sin\gamma(\tau-\pi n) + \theta(\tau-2\pi n)\times \\ \times [A\gamma\sin\gamma(\tau-\pi n) - A\gamma\sin(\tau-3\pi n)] + \\ + \theta(\tau-4\pi n)\,[A\gamma\sin(\tau-3\pi n) - A\gamma\sin(\tau-5\pi n)] + \ldots \qquad (28)$$

It can be easily calculated that at the moments $\tau = 2\pi nk$, $k = 1, 2, \ldots$, the multiplier of $\theta(\tau - 2\pi nk)$ is equal to $u(1 + R)$. Consequently, at the moments of shock the velocity increases by a jump and the increment is given by $u(1 + R)$ in accordance with condition (20).

Differentiating equality (28), we obtain the generalized expression for the acceleration:

$$\ddot{\zeta}(\tau)=\frac{\cos\psi_*^{(0)}}{1-\gamma^2}\cos(\tau+\varphi_*^{(0)})-A\gamma^2\cos\gamma(\tau-\pi n)+$$
$$+\theta(\tau-2\pi n)[A\gamma^2\cos\gamma(\tau-\pi n)-A\gamma^2\cos\gamma(\tau-3\pi n)]+$$
$$+\theta(\tau-4\pi n)[A\gamma^2\cos(\tau-3\pi n)-A\gamma^2\cos\gamma(\tau-5\pi n)]+$$
$$+\ldots+u(1+R)[\delta(\tau-2\pi n)+\delta(\tau-4\pi n)+\ldots] \qquad (29)$$

Here we used again property (18) of the delta-function.

Inserting expression (29) for $\ddot{\zeta}$ into the equations of the first approximation to φ and ψ and repeating literally the reasoning pertaining to it and set forth in Section 36, we find that the values of the phase $\psi_*^{(0)}$ and correction h for the synchronous frequency are determined from Eqs. (18) and (19), Sec. 36. In calculating the functions $P_1(\psi_*^{(0)})$ and $P_2(\psi_*^{(0)})$ note that the integration of these formulas is performed over one period. In the interval from 0 to $2\pi n$, including the moment of shock $\tau=2\pi n$, the acceleration $\ddot{\zeta}(\tau)$, in accordance with formula (29), is given by the expression

$$\ddot{\zeta}(\tau)=\frac{\cos\psi_*^{(0)}}{1-\gamma^2}\cos(\tau+\varphi_*^{(0)})-A\gamma^2\cos\gamma(\tau-\pi n)+$$
$$+u(1+R)\,\delta(\tau-2\pi n) \qquad (30)$$

Substituting this expression into formulas (18), and (19), Sec. 36, and bearing in mind the property (17) of the delta-function, we obtain:

$$\lambda P_1(\psi_*^{(0)})\equiv\int_0^{2\pi n}\ddot{\zeta}(\tau)\cos(\tau+\varphi_*^{(0)})\sin\psi_*^{(0)}\,d\tau=\frac{\pi n}{1-\gamma^2}\sin\psi_*^{(0)}\cos\psi_*^{(0)}-$$
$$-\frac{u^2(1-R^2)}{2}\operatorname{cotan}\varphi_*^{(0)}\tan\psi_*^{(0)}=0 \qquad (31)$$

$$\lambda P_2(\psi_*^{(0)})\equiv\int_0^{2\pi n}\ddot{\zeta}(\tau)\sin(\tau+\varphi_*^{(0)})\cos\psi_*^{(0)}\,d\tau=\frac{u^2(1-R^2)}{2}=2\pi n\lambda h \quad (32)$$

We have limited our treatment to the case of identical unbalances and assumed the parameter p defined in Section 36 to be zero.

Equation (31) determines the generating phase $\psi_*^{(0)}$. One of its solutions is $\psi_*^{(0)}=0$. The other root that is close to $\pi/2$ at small ζ_0 does not satisfy the conditions for the existence of the regime with one shock per period; at $\zeta_0<0$ this root corresponds to the motion without shocks and at $\zeta_0>0$ the body stays pressed by the springs to the stop and there is no motion. The system with zero

clearance, $\zeta_0 = 0$, is an exception. In this case Eq. (31) takes the form

$$\left(1+\frac{2}{\pi n}\cdot\frac{1+R}{1-R}\cdot\frac{f}{1+f^2}\cdot\frac{1}{1-\gamma^2}\right)\sin\psi_*^{(0)}\cos\psi_*^{(0)}=0 \tag{33}$$

It follows that in this special case there are two values that correspond to the synchronous solution, $\psi_*^{(0)} = 0$ and $\psi_*^{(0)} = \pi/2$, and both satisfy the conditions for the existence of the regime with one shock per period.

Equation (32) determines the correction for the synchronous frequency. It has been mentioned in Section 36 that this is the equation of the power balance and in this case this is obvious since the expression $u^2(1 - R^2)/2$ is nothing else than the amount of the kinetic energy lost during the shock, presented in dimensionless form.

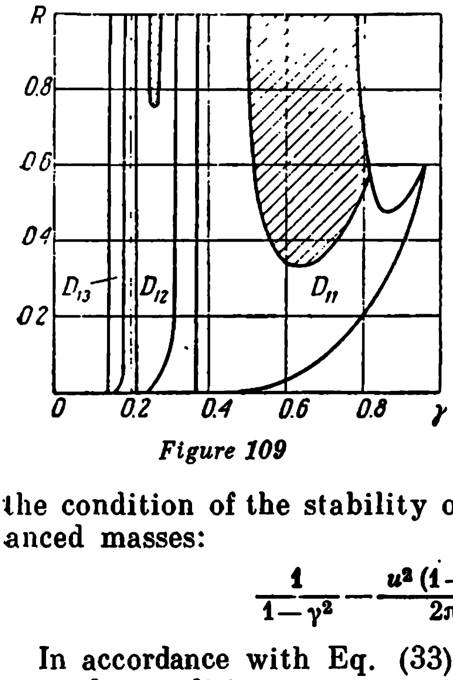

Figure 109

The stability of the synchronous solutions that correspond to the values of the phase $\psi_*^{(0)}$ found is determined, in accordance with what has been stated in Section 36, from the sign of the derivative $dP_1(\psi_*^{(0)})/d\psi_*^{(0)}$. Differentiating with respect to $\psi_*^{(0)}$ the left-hand side of Eq. (31) and substituting into the result $\psi_*^{(0)} = 0$, we obtain the condition of the stability of the in-phase rotation of the unbalanced masses:

$$\frac{1}{1-\gamma^2}-\frac{u^2(1-R^2)}{2\pi n}\cot\text{an}\,\varphi_*^{(0)}<0 \tag{34}$$

In accordance with Eq. (33) at $\zeta_0=0$ condition (34) transforms into the condition

$$\frac{1}{1-\gamma^2}\left(1+\frac{2}{\pi n}\cdot\frac{1+R}{1-R}\cdot\frac{f}{1+f^2}\cdot\frac{1}{1-\gamma^2}\right)<0 \tag{35}$$

The antiphase regime is stable when inequality (35) changes its sign.

Figure 109 shows the graph of the domains of the existence and stability of the periodic motions with one shock per period for a system with $\zeta_0 = 0$; a similar graph was shown in the preceding section. Here the domains of the existence and stability of synchronous in-phase regimes shown for the domains D_{11} and D_{12} have been plotted from inequality (35) (shaded areas). It is characteristic that for the vibration system without shocks whose body has one

degree of freedom there is no synchronization at all at $\gamma < 1$, whereas with shocks there appear domains of in-phase synchronization. The extent of these domains diminishes with increasing n which corresponds to the general properties of nonlinear vibration systems.

As shown in the figure, the in-phase synchronization can be realized only at sufficiently large values of the coefficient of restitution, whereas in the operation of real machines it is generally within the limits 0-0.2. Therefore a shock-and-vibration machine with a single-degree-of-freedom working member, designed after the scheme considered, will not be suited for operation because of the instability of the working in-phase regime.

In a more general case when the working member of the machine has several degrees of freedom and there are more than two unbalances the calculations are carried out in a similar way.

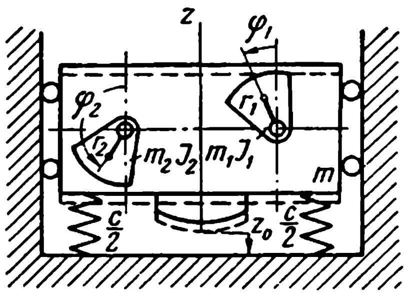

Figure 110

Note that in comparing the shock-and-vibration system with the corresponding shockless system it is found that their equations for the phase $\psi_*^{(0)}$ and the conditions of stability nearly coincide and differ only in that in the former case the expressions contain a shock term. This is confirmed by a comparison of expressions (31) and (34) with formulas (20) and (25), Sec. 36, respectively. Therefore for the system illustrated in Fig. 110 we may directly write the equation in $\psi_*^{(0)}$, the synchronization phase:

$$\left(\frac{1}{1-\gamma_1^2}+\frac{\varepsilon_1\varepsilon_2}{1-\gamma_2^2}-\frac{a^2m}{J}\frac{1}{1-\gamma_3^2}\right)\sin\psi_*^{(0)}\cos\psi_*^{(0)}-$$

$$-\frac{u^2(1-R^2)}{2\pi n}\cot\varphi_*^{(0)}\tan\psi_*^{(0)}=0 \qquad (36)$$

and the condition for the stability of its solution $\psi_*^{(0)}=0$:

$$\frac{1}{1-\gamma_1^2}+\frac{\varepsilon_1\varepsilon_2}{1-\gamma_2^2}-\frac{a^2m}{J}\frac{1}{1-\gamma_3^2}-\frac{u^2(1-R^2)}{2\pi n}\cot\varphi_*^{(0)}<0 \qquad (37)$$

making use of the results of the second example in Section 36. In real machines designed after the scheme in Fig. 110 the values of the parameters in practice always satisfy inequality (37).

It should be pointed out that the stability criterion for synchronous motions formulated at the end of Section 36 is inapplicable in the treatment of shock-and-vibration systems.

43. Power Balance of Shock-and-Vibration Machines

Let us analyse the energy balance of a shock-and-vibration machine using, as an example, an electric vibrohammer (Fig. 111). It consists of the striking part *1*, supported on cap *4* by springs *7* which are located symmetrically with respect to the x-axis. Two symmetrically located identical induction motors *8* with parallel axes are built into the striking part. Unbalanced masses *2* are pressed on the ends of the rotor shafts. The system is centred and the self-synchronization of the unbalanced masses ensures translational vibrations of the striking part along the x-axis. These vibrations alternate with blows of striker *3* on anvil *5* of the cap which is rigidly fixed to the pile *6* being driven into the soil.

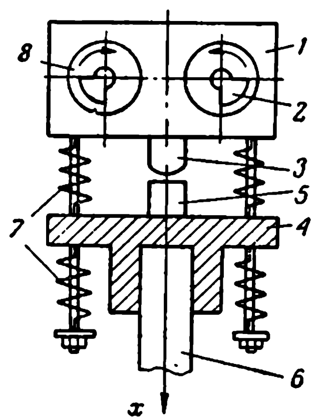

Figure 111

The power N_{el} consumed by the electric motors is spent to compensate for the losses in the kinetic energy of translational motion of the body[1] at shocks (power N_1') and of the motions of the unbalance at shocks (power N_1''); dissipative resistances to the vibration of the body (power N_2); dissipative resistances to the rotation of unbalanced masses (power N_3); additional electrical losses due to jumps in the angular velocity of rotation of unbalanced masses at shocks (power N_4'); additional electrical losses due to variations in the angular velocity of rotation of the unbalanced masses in the intervals between shocks (power N_4''); the usual electrical losses (power N_5).

The power N_1' consumed when the machine is operating with one shock per period is calculated from the expression

$$N_1' = \frac{m_1 (v_-^2 - v_+^2)\,\omega}{4\pi n} \tag{1}$$

where m_1 = mass of the body
v = velocity of body motion
$1/n$ = order of subharmonic vibrations of the body.

Neglecting the dissipative resistances to vibration, the pile mobility and the nonuniformity of rotation of the unbalanced masses and assuming the initial clearance x_0 between striker and anvil to be zero, making use of the velocity relation at shock

$$v_+ = -Rv_- \tag{2}$$

[1] We mean by "body" all the rigidly connected elements of the striking part having translational motion, by "unbalances" the rotating elements (i.e. unbalances proper with their rotors).

where R is the coefficient of restitution of the absolute velocity at impact[1], we obtain the following expression [cf. (6), Sec. 42] for the maximum velocity v_- before the shock:

$$v_- = \frac{2m_0 r\omega}{(1-R)(m_1+m_0)\left(1-\frac{1}{4n^2}\right)} \tag{3}$$

where m_0 = unbalanced masses
ω = mean angular velocity of their rotation
r = eccentricity of unbalanced masses with respect to rotation axes.

Substituting expressions (2) and (3) in equality (1), we obtain

$$N_1' = \frac{(1+R)\,m_1 (m_0 r)^2 \omega^3}{\pi (1-R)\,n\left(1-\frac{1}{4n^2}\right)(m_1+m_0)^2} \tag{4}$$

The power N_1'' is determined from the expression

$$N_1'' = \frac{\omega}{4\pi n}\left[m_0 (u_{x-}^2 - u_{x+}^2) + m_0 (u_{y-}^2 - u_{y+}^2) + J_0 (\omega_-^2 - \omega_+^2)\right] \tag{5}$$

where u_x and u_y = moduli of projections of the velocities of the centres of gravity of unbalances on the x-axis and the y-axis at right angles to it
J_0 = central moment of inertia of unbalances.

With the most severe shocks the radius-vectors r of the unbalanced masses are directed along the y-axis and therefore $u_{y-} = u_{y+} = 0$.

The difference $\omega_-^2 - \omega_+^2$ may be represented approximately by

$$\omega_-^2 - \omega_+^2 = -2\omega\Delta\omega$$

where $\Delta\omega$ is the jump in the angular velocity of the unbalanced masses at shock.

Hence, using formula (8), Sec. 42, we obtain

$$\omega_-^2 - \omega_+^2 = \frac{4\alpha^2 (1+R)\,\omega^2}{(1-R)\left(1-\frac{1}{4n^2}\right)} \tag{6}$$

where

$$\alpha^2 = \frac{(m_0 r)^2}{(J_0+m_0 r^2)(m_1+m_0)} \tag{7}$$

Using this expression and the first of equalities (4), Sec. 42, we may write

$$N_1'' = \frac{(1+R)\,m_0 (m_0 r)^2 \omega^3}{\pi (1-R)\,n\left(1-\frac{1}{4n^2}\right)^2 (m_1+m_0)^2 \left(1+\frac{m_0 r^2}{J_0}\right)} \tag{8}$$

[1] This coefficient coincides with Newton's coefficient if the stop is motionless.

Adding (4) and (8), we obtain

$$N_1 = N_1' + N_1'' = \frac{(1+R)(m_1+\mu m_0)(m_0 r)^2 \omega^3}{\pi(1-R)n\left(1-\frac{1}{4n^2}\right)(m_1+m_0)^2} \tag{9}$$

where

$$\mu = \frac{1}{\left(1+\frac{m_0 r^2}{J_0}\right)\left(1-\frac{1}{4n^2}\right)} \tag{10}$$

We now calculate the power N_2 assuming the dissipative resistance to vibration to be proportional to the absolute velocity of vibration of the body:

$$N_2 = \frac{\omega b}{2\pi n} \int_0^{\frac{2\pi n}{\omega}} v^2\, dt \tag{11}$$

where b is the coefficient of resistance.

Hence

$$N_2 = k v_-^2 b \tag{12}$$

In this expression $k v_-^2$ is the mean square value of the body velocity.

For small damping one may set

$$b = \frac{\vartheta (m_1+m_0)\omega}{2\pi n} \tag{13}$$

in accordance with Table 1 and formulas (29), (43), Sec. 6. Here ϑ is the logarithmic decrement of vibrations.

From expression (13) and equality (3) we obtain

$$N_2 = \frac{2k\vartheta (m_0 r)^2 \omega^3}{\pi(1-R)n\left(1-\frac{1}{4n^2}\right)^2 (m_1+m_0)} \tag{14}$$

The power N_3 can be determined from the formula

$$N_3 = f r_1 m_0 r \omega^3 \tag{15}$$

where f = equivalent coefficient of friction in bearings of unbalances

r_1 = equivalent radius of journals in bearings.

The power N_{mech} spent to compensate for mechanical losses is the sum

$$N_{mech} = N_1 + N_2 + N_3 \tag{16}$$

It is recommended to calculate the power N_4' from the formula

$$N_4' = -\frac{k_1 \Delta\omega}{n\omega} N_{mech} \tag{17}$$

whence, on the basis of equality (8), Sec. 42,

$$N_4' = \frac{2\alpha^2 k_1 (1+R)}{(1-R)\, n \left(1-\frac{1}{4n^2}\right)} N_{mech} \tag{18}$$

Since during the intervals between shocks the vibrohammer is a system operating in the zone beyond resonance (usually far beyond the resonance with shockless operation) it is recommended to calculate the power N_4'' from the formula

$$N_4'' = \frac{k_2 \dot{\varphi}_{2a}}{\omega} N_{mech} \tag{19}$$

where $\dot{\varphi}_{2a}$ is the amplitude of the second harmonic component of torsional vibrations of the unbalance shaft.

In accordance with formula (22), Sec. 33, it can be assumed that

$$N_4'' = \frac{1}{4} \alpha^2 k_2 N_{mech} \tag{20}$$

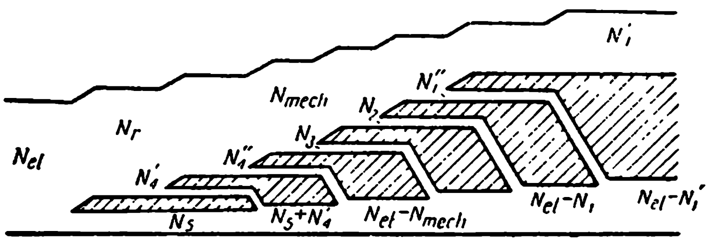

Figure 112

The power N_4 spent to compensate for the additional electrical losses

$$N_4 = N_4' + N_4'' \tag{21}$$

The rated motor power is the sum

$$N_r = N_{mech} + N_4 \tag{22}$$

The power N_5 is calculated from the formula

$$N_5 = \frac{1-\eta}{\eta} N_r \tag{23}$$

where η is the rated efficiency of the motors.

The total active electric power N_{el} consumed by the motors

$$N_{el} = N_r + N_5 \tag{24}$$

and according to formula (23)

$$N_{el} = \frac{N_r}{\eta} \tag{25}$$

The values of the coefficients k, f, k_1, k_2 are based on experience.

Figure 112 shows an approximate example of the power balance of a vibrohammer.

CHAPTER 8

VIBRATORY PROCESSES

44. Influence of Vibration on Dissipative Resistances of the Medium Being Processed

The application of vibration often leads to changes in the character of interaction between the working member of the machine and the external medium as well as in the behaviour of this medium. These changes may arise from seeming as well as from real alterations in the properties of the system, caused by vibrations.

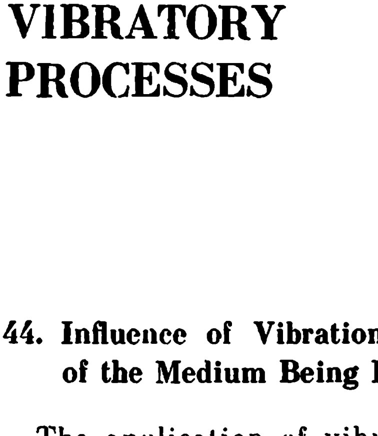

Figure 113

Consider first the effect of vibration on the friction between the working member of the machine and the medium. The apparent reduction in friction under the action of vibration or sliding of one body over another has been known long ago. Suffice it to cite such facts as the self-loosening of nuts in vibrating parts of various machines and the skidding of motor cars in braking to see the importance of the harmful aspect of the problem. These effects have been used with advantage not only in vibration machines but also in some measuring, testing and control devices where, for example, in order to reduce the friction of the shaft in its bearings when the working speed is low the bearings are given a high rotational speed. The most comprehensive theoretical investigation of the coefficient of friction during vibration is due to Blekhman and Janelidze.

Let us study the effect of the relative motion of two rubbing bodies on the apparent (sometimes it is called "effective") coefficient of friction. The body designated by point A slides on another body in the plane of the drawing (Fig. 113). The velocity of the body at this moment is v. The motion may be sustained by some active force of no interest to us or takes place by inertia. Let us impart to body A a momentary impulse in the positive direction of the x-axis; the impulse results in an increment u of the velocity of body

A. At the same moment, $t = 0$, let us apply to body A such a force Q in the direction of u (i.e., in the positive direction of the x-axis) that ensures the constancy of u in the neighbourhood of $t = 0$. It follows that it is necessary to have $Q = F^*$, where F^* is the modulus of the apparent friction force which must be overcome by the force Q.

In Fig. 113, $u = AC$, $v = AB$, $\measuredangle CAB = \varphi$. Let us draw the vector of the resultant velocity $w = AD$. The real friction force F applied to the body A is directed along this vector and opposite to it. Let us resolve the force into two components F^* and F^{**} directed along u and v, respectively. We obtain the following relation:

$$F^* = \frac{u}{w} F \tag{1}$$

Since

$$w = \sqrt{v^2 + u^2 + 2vu \cos \varphi} \tag{2}$$

we can write

$$F^* = \frac{u}{\sqrt{v^2 + u^2 + 2vu \cos \varphi}} F \tag{3}$$

As the friction force is proportional to the coefficient of friction

$$F = fN, \quad F^* = f^* N \tag{4}$$

where f, f^* = real and apparent (effective) coefficients of sliding friction

N = force of normal pressure.

With account taken of equalities (4) expression (3) takes the form

$$f^* = \frac{u}{\sqrt{v^2 + u^2 + 2vu \cos \varphi}} f \tag{5}$$

The angle φ can have any value but since the cosine is an even function it is sufficient to consider φ within the range of angles $0 \leqslant \varphi \leqslant \pi$.

With the following characteristic values of the angle φ the coefficient f^* can be expressed in a particularly simple form:

$$\left.\begin{aligned} f^* &= \frac{u}{v+u} f, \quad (\varphi = 0) \\ f^* &= \frac{u}{\sqrt{v^2 + u^2}} f, \quad \left(\varphi = \frac{\pi}{2}\right) \\ f^* &= \frac{u}{v-u} f, \quad (\varphi = \pi,\ u < v) \end{aligned}\right\} \tag{6}$$

With $u/v \ll 1$ for any angle φ we have

$$f^* = \frac{u}{v} f \tag{7}$$

Here the apparent coefficient of friction and consequently the apparent friction force become proportional to the velocity u. Dry friction becomes, as it were, linearly viscous. That is why the term *linearization of friction* is often used in such cases.

Uniform motion will continue within the limits $0 \leqslant Q \leqslant F$ in the sense that $du/dt = 0$ and with a definite velocity u corresponding to each Q value in accordance with formula (3). If $Q > F$, the motion will be accelerated, i.e., $du/dt > 0$.

Suppose now that the body A in Fig. 113 whose mass is negligibly small performs sinusoidal vibrations in the v direction and its velocity varies according to the law

$$v = v_a \sin \omega t \tag{8}$$

Here the positive direction of vibrations is chosen so to make the angle φ lie within the interval $0 \leqslant \varphi \leqslant \pi/2$. Then, in accordance with (3),

$$Q = \frac{Fu}{\sqrt{v_a^2 \sin^2 \omega t + u^2 + 2v_a u \cos \varphi \sin \omega t}} \tag{9}$$

Solving this equation for the velocity u, we obtain

$$u = \frac{v_a Q^2}{F^2 - Q^2} \left(\sqrt{\cos^2 \varphi + \frac{F^2 - Q^2}{Q^2}} \, |\sin \omega t| + \cos \varphi \sin \omega t \right) \tag{10}$$

It follows that under the action of the force Q the body A performs in the x-axis direction a motion whose velocity is composed of a pulsating (repeated) and an alternating (reversed) components. Since $Q < F$, the swing of the pulsating component is greater than the amplitude of the alternating component and so the velocity u remains non-negative all the time. Twice within the period $2\pi/\omega$ the velocity u becomes zero. It has been assumed in deriving relation (10) that the coefficients of sliding friction and of static friction are equal. The mean velocity u_{mean} of motion of the body A under the action of force Q

$$u_{mean} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} u \, dt$$

The integration of this expression yields

$$u_{mean} = \frac{2v_a Q^2}{\pi (F^2 - Q^2)} \sqrt{\cos^2 \varphi + \frac{F^2 - Q^2}{Q^2}} \tag{11}$$

For the characteristic values of the angle φ we obtain

$$\left. \begin{aligned} u_{mean} &= \frac{2v_a QF}{\pi (F^2 - Q^2)} \quad \text{at} \quad \varphi = 0 \\ u_{mean} &= \frac{2v_a Q}{\pi \sqrt{F^2 - Q^2}} \quad \text{at} \quad \varphi = \frac{\pi}{2} \end{aligned} \right\} \tag{12}$$

In the latter case the alternating component vanishes.

It is of interest to note that, as can be seen from relations (12), in the case of the longitudinally imposed vibration, all other conditions being the same, the mean velocity u_{mean} is greater than in the case of transverse vibration in the ratio $F : \sqrt{F^2 - Q^2}$.

If a force of normal pressure N and an alternating force $\Phi_a \cos \omega t$ directed as N are applied to body *1* (Fig. 114) lying on the plane surface *2*, then on condition that $\Phi_a \leqslant N$ the apparent coefficient of static friction (which is equal to the ratio of the minimum force parallel to plane *2* and disturbing the state of rest to the mean normal pressure N) is determined from the expression

$$f_1^* = f_1 \left(1 - \frac{\Phi_a}{N}\right) \tag{13}$$

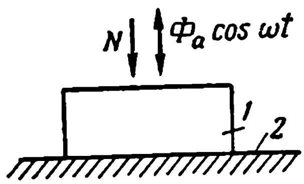

Figure 114

where f_1 is the real coefficient of static friction.

In the cases under consideration the real coefficient of sliding friction was taken to be constant. Therefore the amount of energy dissipated when the body is in a translational motion on the plane without separating from it with any form of the trajectory of total length s is determined from the expression

$$E = \int_0^s f N_i \operatorname{sgn} \dot{s}\, ds \tag{14}$$

where f = real coefficient of sliding friction

N_i = instantaneous value of the resultant of normal pressure forces.

Cases are encountered when under the action of vibration the real coefficient of friction is changed by physicomechanical or physicochemical processes due to vibration, for instance, by the exudation of a liquid phase on the rubbing surfaces.

As a rule, the changes in the coefficient of friction tend to diminish it. The amount of energy dissipated is reduced in such cases and the apparent coefficients of friction decrease more considerably.

It has been observed long since that loose substances consisting of hard particles with the interstices filled by a gas or liquid begin to flow under the action of shaking or jerking. Catastrophic losses in stability of fill dams have been observed, for instance, due to earthquakes and explosions.

In certain cases dry loose substances when subjected to vibration acquire properties similar to the properties of a viscous liquid. In this connection one uses occasionally the phrase "pseudoliquefaction of a loose substance".

Such effects are readily explained by what has been said above about the reduction of the apparent coefficients of friction. The greater the relative velocities of slipping of the particles during vibration, the more pronounced are the effects of the pseudoliquefaction.

Many media possessing only plastic or only elastic properties (or both) at very small rates of shear begin to show viscosity at higher shear rates, i.e., their resistance to deformation becomes dependent on the deformation rate. This kind of nonlinear viscosity whose magnitude depends on the rate of deformation is called *structural viscosity* (in the general case this viscosity is different from zero at zero deformation rate).

The structural viscosity depends only on the rate of deformation at a given moment. If, however, changes in viscosity lag behind the changes in the deformation rate, the term used is thixotropy rather than structural viscosity.

Structural viscosity and thixotropy are, in varying degrees, the properties of such media as colloidal suspensions, soils (especially argillaceous soils where dispersed particles are of colloidal dimensions), concrete mixtures, plastics. The effects of structural viscosity and even of thixotropy are related to the reduction of the apparent coefficient of sliding friction by the fundamental fact that both are caused by the slipping of one body (or one layer of the substance) over another. For such media as not very damp soils and not very mobile concrete mixtures the main role is possibly played by the reduction of the apparent coefficient of friction with imposed vibration.

45. Vibratory Pile Driving and Vibratory Tamping

The aim of the mathematical treatment of vibratory driving and withdrawing of piles is to determine the maximum depth to which the pile can be driven, the time required to reach the given depth, etc. The process of vibratory pile driving is governed by the properties of the soil, a loose or multiphase continuous medium. The mathematical description of these properties at large deformations to which the soil is subjected during the displacement of the pile through it and the solution of the exact equations describing the motion of the pile and the adjacent soil around it would be a very complicated task. Therefore wide use is made of the phenomenological approach to problems of vibratory pile driving. In constructing the dynamic model of the process the pile is considered to be an absolutely rigid bar and the soil is replaced by a system of masses, springs and dampers with dry or viscous friction, selected so as to ensure a satisfactory representation by the model of certain quali-

tative features of the process and quantitative agreement with known experimental data.

Of course, each model can be suited to explain but a limited range of phenomena arising in pile driving. Usually the wider this range the more complex is the model. However, because of the considerable inhomogeneity of soils and a great variety of soil properties, experimental data are very approximate. By virtue of this the requirements in respect of the accuracy of theoretical results are also not very strict. This allows us to be contented with rather simple models giving visible results.

The complete dynamic model of the system of vibratory or shock-and-vibration pile driving must also include the dynamic model of the machine. But the actual mean velocity of pile driving is small as compared to the peak value of vibration velocity of the working member when it is not rigidly fixed to the pile, and the complication of the problem by considering the machine together with the pile, taking into account the approximate character of the theory, is not justifiable.

In the case when the machine body, for example, the body of a pile-driving vibrator is rigidly fixed to the pile, there is no problem of greater complexity of the calculation programme. Thus, in studying the problems of vibratory pile driving one may consider the pile motion under the action of given external forces and the resistance forces of the soil.

The forces of soil resistance to the pile displacement are not constant; they depend on the depth to which the pile has penetrated. Therefore the parameters of the model that represents adequately the actual process must change with penetration. The problem can be radically simplified by making use of the fact that the displacement of the pile during a cycle is much less than the distances over which the change in the resistance forces becomes apparent.

Let us denote the depth to which the pile has penetrated by x, the parameters of the model as functions of x by f_s, $(s = 1, 2, \ldots, n)$. Let us define for the point $x = x_0$ the interval Δx_0 for which the following relation is satisfied:

$$\left| \left(\frac{df_s}{dx} \right)_{x=x_0} \Delta x_0 \right| \ll |f_s(x_0)|, \quad (s=1, 2, \ldots, n) \tag{1}$$

In the interval $x_0 - \Delta x_0/2 < x < x_0 + \Delta x_0/2$ one may, without introducing a large error, replace the parameters f_s by their values at the point x_0, and, solving the equations obtained, find the mean velocity or the mean acceleration of the pile driving within this interval. This simplification is permissible if the absolute pile displacement during the cycle does not exceed the interval Δx_0; this condition is almost always fulfilled in practice.

The authors of most works on the theory of vibratory pile driving usually proceed from the above assumptions. Blekhman and Janelidze made a brief review of the most important works and considered two models of soil resistance: elastic-and-plastic and purely plastic.

Depending on the parameter values of the system, one of the following three types of motion can set in: accelerated motion with a mean acceleration different from zero; regular motion with a certain mean velocity; and purely vibrational motion without penetration. Usually the first regime sets in only at the start of pile driving when the forces of resistance are comparatively moderate. Let us denote by $v(x)$ the mean velocity which characterizes the second regime at a depth of penetration close to x. Let x_{min} be the minimum value of the depth of penetration at which the regular regime sets in; x_{max} the value at which the velocity $v(x)$ vanishes. Evidently, x_{max} is the maximum depth to which the pile can be driven at the given value of the exciting force. If $Q(x)$ is the maximum value of the total force of resistance at the given x and F_{max} is the maximum value of the exciting force $F(t)$, including the constant component, then the quantity x_{max} can be determined from the equation

$$Q(x_{max}) = F_{max} \tag{2}$$

Let the relation between the mean velocity of penetration and depth be known. In this case the lapse of time t_0 for the penetration from $x = x_{min}$ to the limiting depth x_{max} is given by the expression

$$t_0 = \int_{x_{min}}^{x_{max}} \frac{dx}{v(x)} \tag{3}$$

Note that the value of the maximum exciting force F_{max} selected must be based on the bearing capacity of the pile, i.e., on the limiting load that the pile can withstand.

When the pile is driven with the aid of a vibratory pile-driver the force $F(t)$ takes the form

$$F(t) = Mg + P_0 + F_a \cos \omega t \tag{4}$$

where M = mass of pile plus the vibratory pile-driver
P_0 = value of imposed static load
F_a and ω = amplitude and frequency, respectively, of the exciting force developed by the vibratory pile-driver.

Let the driving be carried out with the use of a spring vibrohammer. In accordance with the above assumptions we suppose that the motion of the pile does not affect the motion of the striking part. If c is the stiffness of the supporting springs and $z(t)$ the displacement of the striking part measured from its equilibrium posi-

tion, then the force $F(t)$ in the interval between blows takes the form

$$F(t) = Mg + P_0 + cz(t) \tag{5}$$

The third term representing the vibratory action transmitted to the pile through the springs in the interval between blows in accordance with the results in Section 41 can be written as follows:

$$c\frac{F_a}{m\omega^2}\left[-A\cos\gamma(\omega t-\pi n)+\frac{1}{1-\gamma^2}\cos\omega\bar{\bar{t}}\right] \tag{6}$$

At the moments of shocks $t = 2k\pi/\omega$, $(k = 1, 2, \ldots)$ the pile gets momentary impulse increments whose approximate magnitude is given by the expression

$$M(\dot{x}_+ - \dot{x}_-) = m\cdot\frac{F_a}{m\omega}u(1+R) \tag{7}$$

Using the delta-function introduced in Section 42, we can write the expression for $F(t)$ in the case of shock-and-vibration pile driving in the following form:

$$F(t) = Mg + P_0 + cz(t) + \frac{F_a u(1+R)}{\omega}\left[\delta\left(t-\frac{2\pi n}{\omega}\right)+\right.$$
$$\left.+\delta\left(t-\frac{4\pi n}{\omega}\right)+\ldots\right] \tag{8}$$

We now turn to the consideration of concrete models of the soil. Figure 115 shows a dynamic model representing the main features of the penetration process. Pile *1*, an absolutely rigid bar, is clamped between shoes *2* suspended from springs *3* of stiffness c_1. A force of dry friction whose absolute value is Q_1^0 acts on the pile and shoes. The shoe mass is taken to be zero. This constitutes the model of the forces applied to the lateral surface of the pile. The drag resistance of a similar nature is simulated by spring *4* of stiffness c_2 located under the bottom face of the pile and supported by shoe *5* which overcomes, in sliding over a rough surface, the dry friction force whose absolute value is Q_2^0.

Let us denote the lateral force of resistance by Q_1, and the drag force by Q_2. Figure 116 presents the changes of these forces during the movement of the pile from the moment when all the springs are in their neutral position and the pile bottom face just touches the bottom spring. Spring c_2 is subjected only to compression since the drag resistance is constantly nonpositive. The graph of the displacement shows intervals ($t_1 < t < t_2$, $t_3 < t < t_4$) where no drag resistance is observed.

The equation of motion of the pile may be written as follows:

$$M\ddot{x} = Q_1 + Q_2 + F(t) \tag{9}$$

where

$$Q_1=\begin{cases} -c_1(x-a_i) & \text{at } |x-a_i|<\dfrac{Q_1^0}{c_1} \\ -Q_1^0 \operatorname{sgn} \dot{x} & \text{at } |x-a_i|>\dfrac{Q_1^0}{c_1} \end{cases} \tag{10}$$

$$Q_2=\begin{cases} -c_2(x-x_2) & \text{at } 0<x-b_j<\dfrac{Q_2^0}{c_2} \\ Q_2^0 & \text{at } x-b_j>\dfrac{Q_2^0}{c_2} \\ 0 & \text{at } x-b_j<0 \end{cases} \tag{11}$$

The symbol a_i designates the coordinate of the sequential pile positions in which the force Q_1 is zero after the velocity of the pile on the horizontal part of the graph of the force Q_1 has become zero. The quantity b_j which enters into the definition of the force Q_2 has a similar meaning. In this form Eq. (9) has been given by Blekhman and Janelidze. The alternation of the points a_i and b_j can be traced by referring to the graphs in Fig. 116.

Thus the motion falls into a number of stages and in the transition from one stage to the next the concrete form of the equations changes. The method of fitting reduces the problem to the solution

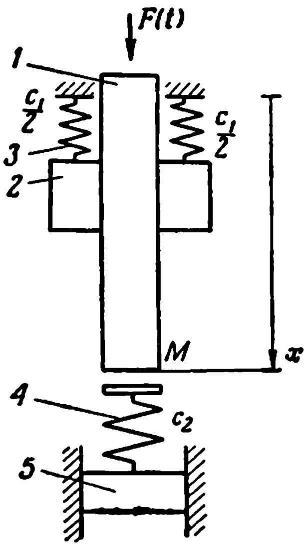

Figure 115

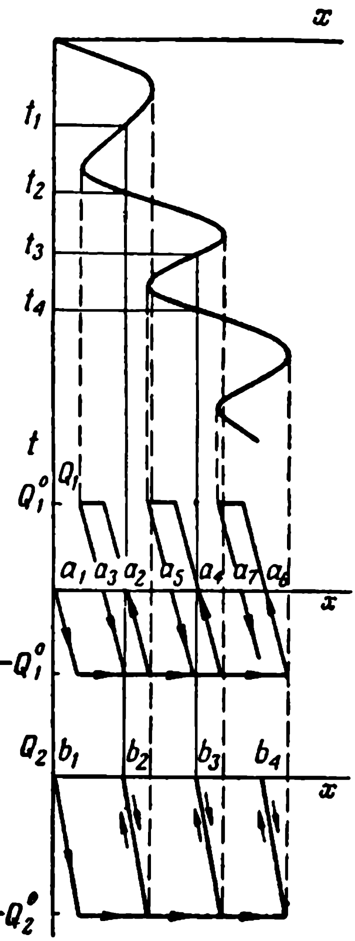

Figure 116

of sets of complicated transcendental equations. The determination of the conditions for the existence and stability of the solutions proves very complicated as well. In the model under consideration even the simplest motion regimes are hardly amenable to analytical treatment. An effective means of studying this system is direct integration of Eq. (9) with the aid of computers.

The problem of vibratory pile withdrawal differs from the above firstly in that the sign of the imposed load P_0 is changed; besides, the shock impulse sign in formula (8) is changed too; secondly the drag force Q_2 does not enter into Eq. (9).

Some data on the character of the pile motion can be obtained without integrating Eq. (9). Blekhman and Janelidze have given in their work sufficient, and in some cases necessary and sufficient, conditions for the existence of accelerated, regular and vibration regimes of motion in vibratory driving and with drawing of piles. It has been established, for instance, that in the case of vibratory pile withdrawing the static effort $-P_0$ must satisfy the inequality

$$P_0 > Q_1^0 + Mg - \frac{c_1 F_a}{|M\omega^2 - c_1|} \tag{12}$$

whereas for purely static withdrawal one must satisfy the inequality $P_0 > Q_1^0 + Mg$. It follows that the imposition of sufficiently intensive vibration results in a considerable reduction of the necessary static force.

The model in which there are no elastic components of resistance forces proves considerably simpler. In this case expressions (10) and (11) for the resistance forces are replaced by the following

$$Q_1 = -Q_1^0 \operatorname{sgn} \dot{x}$$

$$Q_2 = \begin{cases} -Q_2 \text{ at } \dot{x} > 0, & x > x_m \\ 0 \quad \text{at } \dot{x} < 0 \text{ or } x < x_m \end{cases} \tag{13}$$

where x_m is the maximum value of x that has been attained up to the moment considered.

This simplification may be used in many cases, especially with non-cohesive soils.

This model has also been considered by Blekhman and Janelidze. They have obtained the necessary and sufficient conditions for the existence and stability of various regimes of motion. It has been established that the pile can be driven in only if the weight of the installation together with the imposed static load exceeds half the drag force Q_2^0.

If in expression (8) for the force $F(t)$ in shock-and-vibration pile driving the term $cz(t)$ is neglected, which is quite permissible with a small stiffness of the spring suspension, then the problem

of penetrating the soil offering purely plastic resistance becomes elementary. In fact, in this case motion is possible only in one direction, $\dot{x} \geqslant 0$, and Eq. (9) takes the form

$$M\ddot{x} = Mg + P_0 - Q_1^0 - Q_2^0 \tag{14}$$

The pile impulse increment at shock is given by expression (7). Let us denote $(Q_1^0 + Q_2^0 - Mg - P_0)/M$ by $-a$, and $F_a/M\omega$ by $\Delta\dot{x}_0$. If $a > \omega\Delta\dot{x}_0/2\pi n$, then the penetration takes place with stops and its mean velocity

$$V = \frac{\omega(\Delta\dot{x}_0)^2}{2\pi n a} \tag{15}$$

With $a < \omega\Delta\dot{x}_0/2\pi n$ the accelerated regime sets in, whose mean acceleration

$$W = \frac{\omega\Delta\dot{x}_0}{2\pi n} - a \tag{16}$$

A number of works have been devoted to the study of shock-and-vibration pile driving into the soil which offers elastic-and-plastic resistance in the presence of only lateral or only drag resistance. The treatment is usually carried out by the point mapping method with the aid of analogue or digital computers. The practical application of the results obtained from a model of soil resistance is made possible by the availability of reliable experimental data. In some cases, especially in driving long piles, the wave character of strain propagation through the pile begins to play a considerable role and the models treated above prove inapplicable.

Stress should be laid on the fundamental difference between vibratory and shock-and-vibration pile driving (or withdrawing). Vibratory driving with the aid of a vibration generator rigidly fixed to the pile is feasible only in cases when a sufficiently large constant component of the forces is applied to the pile in the direction of penetration. Such a component can appear even in the absence of weight or other static forces.

In this case its appearance is due to anisotropic resistance forces, i.e., when the resistance to motion in one direction exceeds, on the average, the resistance in the opposite direction. In vibratory pile driving (or withdrawing) the directed displacement is effected by the action of this constant component. Vibration either reduces the apparent or real resistances to motion or realizes the effects of the anisotropic force of resistance.

Shock-and-vibration pile driving with the aid of a vibration generator connected with the pile through springs and dealing it blows with the generator body is feasible in the absence of a constant component of the forces which are applied to the generator-pile

system and even in a direction opposite to the constant component (provided its value is not too large). The necessary condition for shock-and-vibration driving is the presence of sufficient forces of friction between pile and soil. In shock-and-vibration pile driving (or withdrawing) this friction is actually the driving force. The shock-and-vibration motion of the vibration generator gives rise to a dissymmetry of the resultants of forces applied to the pile and makes the friction the driving force. Similar results could be produced by a vibration machine rigidly fixed to the pile and exciting a nonsinusoidal force if the difference between its maximum and minimum moduli were sufficiently large.

The problem of idealization of the vibratory tamping process has much in common with that of the vibratory and shock-and-vibration processes of pile driving since in all these cases we have to consider the interaction between an absolutely rigid body and a continuous medium—the soil. The efficiency of the purely vibratory process of soil compacting when the working member of the soil compacting machine is continuously in contact with the soil is comparatively low. Therefore in the following we shall concern ourselves only with the shock-and-vibration method of soil compacting.

The motion of the working member of a shock-and-vibration soil compacting machine consists of two stages: the motion in the air and the motion in the soil. The simplest idealization of the latter stage was already used in Section 41 where the motion of a shock-and-vibration tamper with momentary shocks against an undeformable stop was studied. This scheme is suitable only for the description of the operation of machines on very rigid foundations. But in the general case the assumption of momentary interaction with the soil proves to be too rough and leads to a considerable discrepancy between theory and experiment.

The idealized process of soil compaction must, firstly, describe the changes in the physical properties of the soil during tamping and furnish an objective estimate of the quality of compacting. Secondly, it must enable correct equations to be obtained for the motion of the vibrotamper. The first requirement can be consistently satisfied by considering the soil to be a continuous medium possessing elastic-and-plastic properties. This approach permits one to find the changes in the density of the soil after a shock wave has passed through it, and to estimate the dimensions of the region over which the compaction is effected and the number of blows necessary to obtain the required effect. An example of such an approach correlated with the experimental data can be found in the paper by S. Grigorian.

However in studying the dynamics of the tamping machine this approach involves considerable difficulties: during the contact

the equation of motion of the machine is the boundary condition for the equation describing the wave propagation in the soil. In this case the description of the soil by a dynamic model is much more convenient; such an approach is similar to that used in the treatment of vibratory pile driving. Of course, the kind of model and its parameters must correspond to the picture of the compacting process obtained by precise treatment and experiment. The vibrotamper can be idealized as a rigid body performing vibrations under the action of a sinusoidal exciting force. The vibratory motion of the vibrotamper in the vertical direction is usually accompanied by a horizontally directed displacement at a mean velocity different

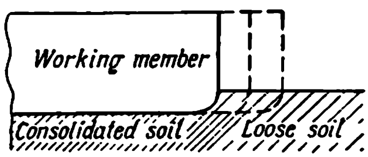

Figure 117

from zero. For the sake of simplicity we shall consider the compaction of a horizontal surface; the consideration of the compaction of slopes does not yield anything new in principle. The tamping proceeds in much the same manner as shown in Fig. 117; before each blow the working member is displaced somewhat towards the soil not yet compacted and is therefore always under the same conditions, with the ground properties remaining unchanged.

The dynamic model adequately representing this process is pictured in Fig. 118. Since the dimensions of the working member *1* in plan are many times the depth of subsidence per blow it is natural to neglect the forces of lateral resistance of the soil and take into account only the drag forces. The drag is simulated by spring *3* of stiffness c_2 which presses against shoe *4* held on the guiding surfaces by the force of dry friction Q_2^0. Let us denote the mass of the working member by M and direct the x-axis vertically upward; the motion of the working member will be described by the equation

$$M\ddot{x} = -Mg + Q_2 + F_a \cos \omega t \qquad (17)$$

which differs from Eq. (9) in the direction of the x-axis and also in that $P_0 = 0$ and $Q_1 = 0$. The force Q_2 is given by the expression

$$Q_2 = \begin{cases} -c_2(x - b_j) & \text{at } 0 > x - b_j > -\dfrac{Q_2^0}{c_2} \\ Q_2^0 & \text{at } x - b_j < -\dfrac{Q_2^0}{c_2} \\ 0 & \text{at } x - b_j > 0 \end{cases} \qquad (18)$$

which differs from (11) only in sign. The quantity b_j takes up two values: $b_0 = 0$ before the beginning of contact and at first stage of joint motion; $b_1 = x_m + Q_2^0/c_2$ after the resistance is a plastic force; x_m is the maximum downward displacement of the working member. After each blow the system simulating the soil returns to its initial state as shown schematically in Fig. 118. The graph in Fig. 119 is an approximate example of the displacement of the vibrotamper; the heavy line represents the motion in the soil and the thin one that in the air.

Attempts have been made in some cases to take into account the soil mass drawn into motion. For this purpose a variable mass

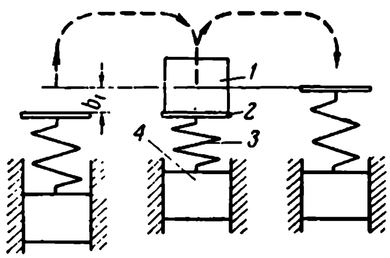

Figure 118

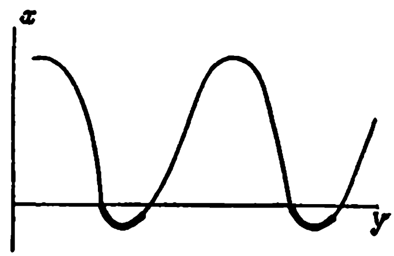

Figure 119

value is assigned to flat piece *2* fixed to the upper end of spring *3*; the mass is proportional to the downward displacement of the working member measured from the surface $x = 0$ and is called the *reduced mass*. This modification of the scheme makes the equations of motion nonlinear for the shoe being in contact with the soil. Clearly this complicates the problem considerably. There are some other dynamic models simulating the properties of the soil. However, one can, by a suitable choice of parameters, arrive at an agreement between the results provided by any verisimilar scheme and experimental data.

Therefore the following method of taking into account the influence of the soil properties on the motion of the vibrotamper may be suggested. We shall be interested only in the final result of the interaction between the working member and the soil without considering the process of interaction proper. This result can be characterized by simultaneously specifying three quantities: the duration of interaction Δt and the position and velocity of the working member at the moment of separation from the soil. All the three parameters lend themselves readily to determination by experiment. Note that the coordinate of the working member at the moment of separation is b_1. This is usually a small quantity as compared to the vibration swing of the working member performing its motion in the air; without introducing large errors it may be set equal to zero. The velocity after separation can be conveniently specified by

expressing it in terms of the velocity at the initial moment of interaction. If t_0 is the initial moment of interaction, we can assume that

$$\dot{x}(t_0+\Delta t) = -R\dot{x}(t_0) \tag{19}$$

Since $\dot{x}(t_0) \leqslant 0$ and $\dot{x}(t_0+\Delta t) > 0$ at all times the coefficient R is non-negative. It is similar to Newton's coefficient of velocity restitution in the classic impact theory and is reduced to it when $\Delta t \to 0$. The analogy is not exhaustive. Actually the limiting condition $R \leqslant 1$ in this case does not seem logically indispensable. The experimental values of R, however, always satisfy this condition.

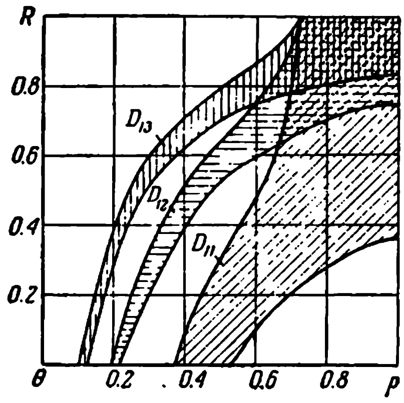

Figure 120

Thus we have been led to the following idealization of the working process of the tamping machine. With $x > 0$, its motion is described by the equation

$$M\ddot{x} = -Mg + F_a \cos \omega t \tag{20}$$

Having come, at a certain moment $t = t_0$, into contact with the surface of the stop, $x = 0$, with the velocity $\dot{x}(t_0)$ the machine separates from the surface after the time interval Δt with a velocity given by the relationship (19). In terms of the dimensionless parameters (4), Sec. 41, Eq. (20) coincides with Eq. (31), Sec. 41, and the condition (19) takes the form

$$\dot{\zeta}(\tau_0+\Delta\tau) = -R\dot{\zeta}(\tau_0) \tag{21}$$

where $\Delta\tau = \omega\Delta t$.

The dynamics of this system have been studied by A. Dorokhova and S. Lukomsky. The character of the periodic motions and the picture of their alternation in the parameter space remain the same as for the system with momentary shocks, but the extent and location of the domains of the existence and stability are changed. Figure 120 shows the domains D_{11}, D_{12}, D_{13} in the plane of the parameters p, R for $\Delta\tau = \pi/2$. A comparison of this graph with Fig. 105 for $\Delta\tau = 0$ shows that with the duration of interaction taken into account we have a displacement of the boundaries and a general extension of the domains of the existence and stability.

The results of such a treatment allow one to choose properly the vibrotamper parameters, provided the coefficients R and $\Delta\tau$ have been correctly selected. In other schemata of shock-and-vibration machines, similar results are obtained if the duration of interaction is taken into account. The investigation of the shock-and-vibration

system illustrated in Fig. 98 and carried out for the case $\Delta\tau \neq 0$ can serve as an example.

The operating conditions of the tamping machine must ensure high shock velocities and must not be affected by changes in the parameters of the working medium—the soil—since these changes may be large. In this respect the simplest schema in Fig. 104 is not the best one: the domains of the existence of subharmonic regimes prove too narrow. Better results are ensured by the schema of the usual vibrohammer in Fig. 98 if used as a tamper.

The self-translation of the tamping machine over the surface being compacted is not considered here. Between this problem and the problem of the motion of a material particle over a rough vibrating plane surface there is an analogy, as has been pointed out by Blekhman and Janelidze, which permits one to apply the results of the theory of vibratory conveying to the description of the motion of a vibrotamper.

46. Vibratory Conveying

Of all the technological processes based on the use of vibration the vibratory conveying process has been the most comprehensively covered by theoretical studies. This is due, to a certain extent, to the relatively simple mathematical model of the phenomenon. The idealization of the process of conveying has much in common with the description of other processes where the motion directed on the average is generated or sustained by the action of vibration.

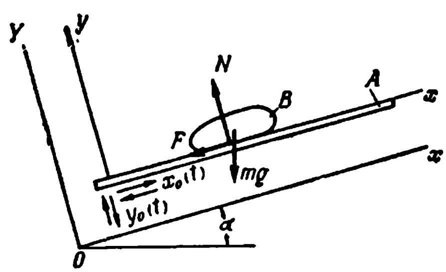

Figure 121

The present state of the theory of vibratory conveying is thoroughly presented in the book *Vibratory Displacement* by Blekhman and Janelidze. The book also sets out the theory of the working processes of some vibration machines based on the vibratory displacement process. In view of this we shall consider here only the formulation of the problem and cite a number of practically important results of the theory.

The dynamic model of the process is illustrated in Fig. 121. The plane A is inclined at an angle $-\pi/2 < \alpha < \pi/2$ to the horizontal plane. The OY-axis of the fixed system of coordinates OXY is perpendicular to the plane A; the OX-axis lies in the plane A when the latter is not moving. A moving reference system O_1xy is attached to the plane A that performs translational vibrations in the OXY plane according to the law $X = x_0(t)$, $Y = y_0(t)$; the axes of the O_1xy system are parallel to the fixed axes. Consider a heavy material

particle B of mass m having a motion of translation in the OXY plane. Its absolute coordinates X, Y are related to the relative coordinates x, y by the expressions

$$X = x + x_0(t), \quad Y = y + y_0(t) \tag{1}$$

If the particle is not in contact with the plane A, it is acted upon only by the force of gravity whose components in the directions of the OX- and OY-axes are $-mg \sin \alpha$ and $-mg \cos \alpha$, respectively. While in contact with the plane the particle B is acted upon by the reaction whose normal component will be denoted by N. We assume that the tangential component F is of the nature of dry friction. It is this friction that is supposed to be responsible for the direction of motion on the average. It follows that the equations of motion of the particle take the form

$$\left.\begin{aligned} m\ddot{X} &= -mg \sin \alpha + F \\ m\ddot{Y} &= -mg \cos \alpha + N \end{aligned}\right\} \tag{2}$$

Substituting (1) in Eqs. (2), we obtain the equations of motion of the particle with respect to the vibrating plane A:

$$\left.\begin{aligned} m\ddot{x} &= -m\ddot{x}_0(t) - mg \sin \alpha + F \\ m\ddot{y} &= -m\ddot{y}_0(t) - mg \cos \alpha + N \end{aligned}\right\} \tag{3}$$

which contains the last terms only in the case when $y = 0$. This condition yields

$$N = m\ddot{y}(t_0) + mg \cos \alpha \tag{4}$$

The force of dry friction is given in this case by the expression

$$F = -fN \operatorname{sgn} \dot{x} \tag{5}$$

where f is the coefficient of sliding friction.

The particle can be on the plane also in a state of relative rest when $x = 0$. From the first of Eqs. (3) we obtain that in this case

$$F = F_0 = m\ddot{x}_0(t) + mg \sin \alpha \tag{6}$$

The value of the force of static friction F_0 cannot be arbitrary. It is limited by the condition

$$|F_0| < f_1 N \tag{7}$$

where f_1 is the coefficient of static friction. If this condition is violated, the particle begins to slip.

Thus, we have established that the particle can be in one of the three following states of relative motion:

(1) the state of flying described by the equations

$$\left.\begin{aligned}\ddot{x} &= -\ddot{x}_0(t) - g\sin\alpha \\ \ddot{y} &= -\ddot{y}_0(t) - g\cos\alpha\end{aligned}\right\} \tag{8}$$

(2) the state of relative slipping the equation of which is obtained upon substituting relations (4) and (5) into the first of Eqs. (3):

$$\ddot{x} = -\ddot{x}_0(t) - g\sin\alpha - f[\ddot{y}_0(t) + g\cos\alpha]\operatorname{sgn}\dot{x} \tag{9}$$

(3) the state of relative rest.

We assume that in the transition from flying to relative slipping the normal component of the relative velocity, $\dot{y}$, changes in accordance with Newton's theory of impact

$$\dot{y}_+ = -R\dot{y}_- \tag{10}$$

and the tangential component changes only in magnitude, but not in direction, its values before and after the impact being related by the expression

$$\dot{x}_+ = (1-\lambda)\dot{x}_- \tag{11}$$

Here λ is the so-called coefficient of momentary friction upon impact; its values are within the segment $0 \leqslant \lambda \leqslant 1$.

The relations (6) through (11) determine completely the motion of the particle. Each stage of motion is described by differential equations of a simple form [it will be recalled that the functions $x_0(t)$ and $y_0(t)$ have been specified]. The sequence of these stages in different order yields a great variety of possible motion regimes. A certain definite kind of motion sets in after some time, this motion being independent of the initial conditions. It is the study of such steady motions that constitutes the subject matter of the theory.

All steady motion regimes can be divided into two groups: regular regimes where the particle velocity along the plane is a periodic function of time, and accelerated motion regimes that are characterized by a constant mean acceleration along the plane. Thus, the regular regimes are described by relations of the form

$$x(t) = Vt + \varphi(t), \quad y = \psi(t) \tag{12}$$

whereas the solution of the motion equations corresponding to the accelerated motion regimes can be presented in the following form:

$$x(t) = \frac{Wt^2}{2} + V^*t + \varphi_1(t), \quad y = \psi_1(t) \tag{13}$$

The quantities V, W, V^* are constants, the functions φ, φ_1, ψ and ψ_1 are periodic with a period equal to or a multiple of the period of the functions $x_0(t)$ and $y_0(t)$.

In the cases most important for practical applications the plane A vibrates according to the harmonic law

$$\left.\begin{aligned} x_0(t) &= a\cos\omega t \\ y_0(t) &= b\cos(\omega t-\varepsilon) \end{aligned}\right\} \tag{14}$$

At $\varepsilon = 0$ all points of the plane vibrate rectilinearly at an angle $\beta = \tan^{-1}\frac{b}{a}$ to the O_1x-axis at the amplitude $A_0 = \sqrt{a^2+b^2}$. With $\varepsilon = \pi/2$ and $a = b$ each point of the plane moves in circular and in all other cases in elliptical trajectories.

Consider first the motion without separation from the vibrating plane. For this motion to be realized it is necessary that the normal reaction N be positive at all times. Substituting $y_0(t)$ from the second of equalities (14) into expression (4) it is found that the condition for motion without separation takes the form

$$\frac{g\cos\alpha}{b\omega^2} > 1 \tag{15}$$

Upon introducing dimensionless variables

$$\xi = \frac{x}{a}, \quad \tau = \omega t \tag{16}$$

and substituting relations (14) into Eq. (9) of the motion without separation the latter is transformed into the form

$$\ddot{\xi} = \cos\tau - G + \mu\,[\cos(\tau-\varepsilon) - \Gamma]\operatorname{sgn}\dot{\xi} \tag{17}$$

The dimensionless parameters that enter into this equation have the following values:

$$G = \frac{g\sin\alpha}{a\omega^2}; \quad \Gamma = \frac{g\cos\alpha}{b\omega^2}; \quad \mu = f\frac{b}{a} \tag{18}$$

The dot denotes now differentiation with respect to the dimensionless time τ.

One can ascertain by simple reasoning that the accelerated motion regime can exist only at $|G| > \mu\Gamma$ or, reverting to the original notations, at

$$|\tan\alpha| > f \tag{19}$$

Thus, a motion accelerated on the average cannot in general set in on a horizontal plane surface. This fact is self-evident.

Depending on the values of the parameters regular regimes of various types can be established in the system. All of them have been studied in detail in the above-mentioned book by Blekhman and Janelidze for the case $\varepsilon = 0$. We shall restrict our treatment to a partial consideration of only one of the possible regular regimes with two momentary stops per period equal in dimensionless units

to 2π. With this regime of motion the particle slides in the positive O_1x-axis direction during one part of the period and in the opposite direction during another. In this case there are no intervals of relative rest.

Let us denote the initial moments of the "forward" and "back" motion by τ_0 and τ_1, respectively. Then $\dot{\xi} > 0$ at $\tau_0 < \tau < \tau_1$ and $\dot{\xi} < 0$ at $\tau_1 < \tau < 2\pi + \tau_0$. At the moments of time τ_0, τ_1 and $2\pi + \tau_0$ the velocity $\dot{\xi}$ becomes zero.

Within the interval $\tau_0 < \tau < \tau_1$ the motion is described by the equation

$$\ddot{\xi} = \cos\tau + \mu\cos(\tau - \varepsilon) - G - \mu\Gamma \tag{20}$$

Its solution, which becomes zero at $\tau = \tau_0$, can be written as follows:

$$\dot{\xi} = \sin\tau - \sin\tau_0 + \mu\sin(\tau - \varepsilon) - \mu\sin(\tau_0 - \varepsilon) - (G + \mu\Gamma)(\tau - \tau_0) \tag{21}$$

This expression must become equal to zero at $\tau = \tau_1$ as well:

$$\sin\tau_1 - \sin\tau_0 + \mu\sin(\tau_1 - \varepsilon) - \mu\sin(\tau_0 - \varepsilon) - (G + \mu\Gamma)(\tau_1 - \tau_0) = 0 \tag{22}$$

Similarly the equation of motion within the interval $\tau_1 < \tau < 2\pi + \tau_0$ is

$$\ddot{\xi} = \cos\tau - \mu\cos(\tau - \varepsilon) - G + \mu\Gamma \tag{23}$$

Its solution takes the form

$$\dot{\xi} = \sin\tau - \sin\tau_1 - \mu\sin(\tau - \varepsilon) + \mu\sin(\tau_1 - \varepsilon) - (G - \mu\Gamma)(\tau - \tau_1) \tag{24}$$

and becomes zero at $\tau = 2\pi + \tau_0$:

$$\begin{aligned}\sin\tau_0 - \sin\tau_1 - \mu\sin(\tau_0 - \varepsilon) + \mu\sin(\tau_1 - \varepsilon) -\\ -(G - \mu\Gamma)(2\pi + \tau_0 - \tau_1) = 0\end{aligned} \tag{25}$$

Equations (22) and (25) determine the moments of transition from one stage of motion to the other. Introducing the notations

$$\frac{\tau_1 - \tau_0}{2} = \gamma; \qquad \frac{\tau_1 + \tau_0}{2} = \delta \tag{26}$$

and making simple transformations, we get

$$\left.\begin{aligned}\sin\gamma\cos\delta &= G\gamma - \frac{\pi}{2}(G - \mu\Gamma)\\ \mu\sin\gamma\cos(\delta - \varepsilon) &= \mu\Gamma\gamma + \frac{\pi}{2}(G - \mu\Gamma)\end{aligned}\right\} \tag{27}$$

With $\varepsilon=0$ the solution of the equations is straightforward:

$$\left.\begin{aligned} \gamma &= \frac{\pi(1+\mu)}{2\mu}\cdot\frac{\mu\Gamma-G}{\Gamma-G} \\ \cos\delta &= \frac{G\gamma-\frac{\pi}{2}(G-\mu\Gamma)}{\sin\gamma} \end{aligned}\right\} \tag{28}$$

In particular, for the motion over a horizontal plane:

$$\left.\begin{aligned} \gamma &= \frac{\pi}{2}(1+\mu) \\ \cos\delta &= \frac{\pi\mu\Gamma}{2\sin\frac{\pi(1+\mu)}{2}} \end{aligned}\right\} \tag{29}$$

Hence, the condition for the existence of the regime being considered at $G=0$ is [the left inequality is the condition (15)]:

$$1\leqslant\Gamma\leqslant\frac{2}{\pi\mu}\sin\frac{\pi(1+\mu)}{2} \tag{30}$$

In the general case ($\varepsilon\neq 0$, $G\neq 0$) the solution of Eqs. (27) and the condition for the existence of the regime take a somewhat less simple form.

An additional restriction follows from the condition that there are no pauses of finite duration. Referring to Eq. (9) it is found that at the moments of transition the following inequality must be fulfilled:

$$|\ddot{x}_0(t)+g\sin\alpha|\geqslant f[\ddot{y}_0(t)+g\cos\alpha]$$

or, using the above notations,

$$|\cos(\delta-\gamma)-G|\geqslant\mu[\Gamma-\cos(\delta-\gamma-\varepsilon)] \tag{31}$$

$$|\cos(\delta+\gamma)-G|\geqslant\mu[\Gamma-\cos(\delta+\gamma-\varepsilon)] \tag{32}$$

Our main task is the determination of the mean velocity at which the particle is conveyed. It can be expressed as follows:

$$\dot{\xi}_{mean}=\frac{1}{2\pi}\left(\int_{\tau_0}^{\tau_1}\dot{\xi}\,d\tau+\int_{\tau_1}^{2\pi+\tau_0}\dot{\xi}\,d\tau\right) \tag{33}$$

Into the first and second integrals on the right-hand side of the last equation we substitute expressions (21) and (24) for ξ, respectively. Having made the calculations and taking into account relations (27), we obtain the mean speed of the conveying:

$$\dot{\xi}_{mean}=-\cos\gamma\sin\delta-\frac{2\mu}{\pi}\left[\left(\gamma-\frac{\pi}{2}\right)\cos\gamma-\sin\gamma\right]\sin(\delta-\varepsilon) \tag{34}$$

where the quantities τ_0 and τ_1 have been replaced by their values from expressions (26).

In most of the vibratory conveying equipment used at present the working member performs rectilinear oscillations ($\varepsilon = 0$) which is partly explained by the simplicity of the drive ensuring such a motion. The investigations carried out by V. Yakubovich and R. Brumberg have shown that considerably higher velocities can be obtained with elliptic vibrations of the working member than in the case of rectilinear vibrations. Indeed from expression (34) for the mean velocity, viewed as a function of ε, it does not follow that (34) has an extreme value at $\varepsilon = 0$. The direct determination of the extreme values of function (34) is rather difficult since its arguments γ, δ and ε are not independent but related by expressions (27). The graph in Fig. 122 illustrates the relation between the mean velocity $\dot{\xi}_{mean}$ and the angle ε for $\mu = 0.05$; $\Gamma = 1$. This graph has been plotted with the aid of a continuous-action computer by simulating Eq. (17).

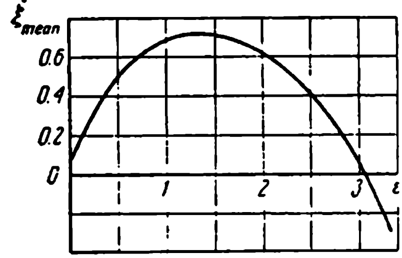

Figure 122

The study of the relation between the velocity of conveying and the angle ε is in fact a special case of the problem of determining the optimum law governing the vibration of the working member so as to ensure the maximum velocity of the conveying. In other words, it can be reduced to the determination of the required form of the functions $x_0(t)$ and $y_0(t)$.

This formulation of the problem requires a more precise statement of a number of points. First of all, in order to make various vibration laws comparable we shall assume the functions $x_0(t)$ and $y_0(t)$ to have the following form:

$$x_0(t) = a\Phi_1(\omega t), \quad y_0(t) = b\Phi_2(\omega t) \tag{35}$$

where $\Phi_1(\omega t)$ and $\Phi_2(\omega t)$ are dimensionless periodic functions of their argument which are normalized according to the conditions

$$|\Phi_1(\omega t)|_{max} = 1, \quad |\Phi_2(\omega t)|_{max} = 1 \tag{36}$$

Suppose we have succeeded in obtaining the solution of the set of Eqs. (3) that describe completely the motion, the functions $x_0(t)$ and $y_0(t)$ being given in the general form (35). We now construct an expression for the mean velocity of conveying similar to (34). It will depend on the form of the functions Φ_1 and Φ_2, i.e., will be the functional of Φ_1 and Φ_2. The finding of the functions Φ_1 and Φ_2 with which this functional reaches its maximum value is a typical problem of the calculus of variations.

This direct approach cannot be realized since it is impossible to solve in the general form the transcendental equations determining the various moments of transition from one stage of motion to the next, and also because of the necessity of taking into account the conditions for the existence and stability of various regimes. Besides, very strong restrictions are often imposed on the required result, though they are not implied in the mathematical essence of the problem. For example, one of the very important requirements is that the functions $x_0(t)$ and $y_0(t)$ must be simply and economically realizable with the aid of up-to-date technical means. Restrictions are also imposed on the form of the working regime. For instance, if it is required that the process be as far from noisy as possible, the motion must take place under the regime without separation. Under these conditions the problem acquires much greater definiteness. In the problem considered we dealt with restrictions of the same kind: the functions $\Phi_1(\omega t)$ and $\Phi_2(\omega t)$ were represented by the functions $\cos \omega t$ and $\cos(\omega t - \varepsilon)$. Since the form of the functions Φ_1 and Φ_2 was predetermined, the variational problem was reduced to the simple determination of the extremum.

Usually, in seeking the optimum vibration law for the working member of a vibratory conveyer one postulates the class of permissible functions Φ_1 and Φ_2 and the kind of regime. Even in this case the purely analytic solution process proves difficult or even impossible. Therefore, to obtain the solution one resorts to graphic or graphical-analytical methods and also to the use of computers. The last method is especially convenient as the solution of the equations of motion is obtained directly and there is no need to postulate the kind of regime and to take into account the conditions of its existence and stability. The variational problem is solved by going over the variants.

Let us revert to the process of conveying without separation over a horizontal plane surface. Clearly, if the vibrations of the plane are symmetrical with respect to the vertical axis, there will then be no motion which is directed on the average[1]. In the above-discussed case of the vibrations of the plane points along elliptical trajectories the asymmetry was attained by the inclination of the ellipse axes with respect to the coordinates axes; with rectilinear vibrations their direction was at an angle β to the plane surface. Another method of exciting a directed movement of particles is also feasible, in which there are no vertical vibrations and the plane surface vibrates in the horizontal direction according to an unsymmetric law. With the above notations

$$b=0,\ G=0,\ x_0(t)=a\Phi_1(\omega t) \tag{37}$$

[1] Except in the case of anisotropic friction mentioned in Section 44.

The question arises now how to find the optimum law for $\Phi_1(\omega t)$. Agranovskaya and Blekhman solved the problem formulated as follows.

Assume the following class of permissible functions of the form

$$\Phi_1(\omega t) = -\frac{1}{A_m}\left[\cos\omega t - \frac{\rho}{4}\cos(2\omega t + \varepsilon)\right] \qquad (38)$$

where ρ = ratio of the acceleration amplitude of the second harmonic to that of the first

ε = phase difference between the harmonics

$A_m > 0$ = normalizing coefficient ensuring the fulfillment of condition (36).

Such a vibration law can be realized, for example, by using a special vibration generator having four shafts. We now introduce the quantity

$$k_f = \frac{a\omega^2 \,|\,\Phi_1''(\omega t)\,|_{max}}{gf} \qquad (39)$$

which is the ratio of the maximum plane surface acceleration to the acceleration of gravity multiplied by the coefficient of friction. This quantity characterizes the dynamic loads in the drive of the vibratory conveyer. It is required to select such values of the parameters ρ and ε that would ensure the maximum velocity of conveying at the given value of the overload coefficient k_f.

The solution was obtained with the aid of an analog computer by going over the variants. The results had the form of a graph of ρ_{opt} and ε_{opt} versus k_f (see Fig. 123).

This problem has also been solved by Troitsky, but for another class of permissible functions. Limiting his treatment to piecewise-continuous laws of variation of the plane acceleration Troitsky demonstrated that the optimum law ensuring the maximum mean particle velocity at the given overload coefficient is determined by the relations

$$a\omega^2\Phi(\omega t) = \begin{cases} -W_{max} & \text{at } 0 < t < \left(1 + \frac{1}{k_f}\right)\frac{\pi}{2\omega} \\ W_{max} & \text{at } \left(1 + \frac{1}{k_f}\right)\frac{\pi}{2\omega} < t < \frac{\pi}{\omega} \\ gf & \text{at } \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases} \qquad (40)$$

Figure 124 illustrates the relation between the mean velocity of conveying and the overload coefficient k_f for various laws of vibration of the working member. Curves *1* and *2* refer to two existing types of vibratory conveyers with biharmonic laws of vibration of the working member; curve *3* corresponds to the optimum biharmonic law of vibration; finally, curve *4* corresponds to the piecewise-continuous law (40).

Referring to the graph in Fig. 124 one can see that the use of optimum forms of vibration results in a considerable increase in the mean velocity, this increase being the greater the higher the given overload coefficient.

It has been found that the optimum biharmonic law is closely approximated by the two first harmonics of the expansion of the piecewise-continuous law (40) in a Fourier series. It follows that the considerable increase in the mean velocity provided by the piecewise-continuous vibration law as compared to the optimum

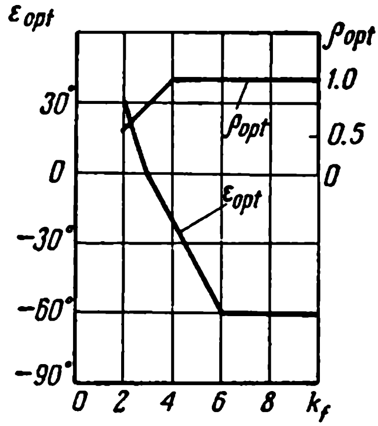

Figure 123

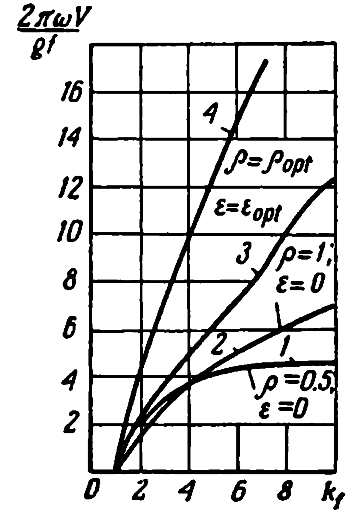

Figure 124

biharmonic law is due to the presence of higher harmonics. However, the practical realization of the vibration law (40) involves serious technical difficulties.

We have so far treated only motions without separation for which condition (15) is fulfilled. If this condition is not satisfied, i.e., $\Gamma < 1$, the particle in its motion separates from the vibrating plane surface. Two cases are to be distinguished here. At $R = 0$ all the three types of motion are possible: the flight of the particle, its relative slipping, and its relative rest. With $R \neq 0$ theoretically only one type of motion—the particle flight—is possible, the stages of flight alternating with momentary impacts on the vibrating plane. A characteristic feature of motion with tossing is the great variety of steady regimes. The most important regimes have been discussed in the book by Blekhman and Janelidze.

With $R \neq 0$ the problem of conveying is reduced to the problem of the motion of a springless shock-and-vibration system having one degree of freedom (see Section 41). Let us assume the functions

$x_0(t)$ and $y_0(t)$ to have the form (35) and introduce the dimensionless variables

$$\xi = \frac{x}{a}, \quad \zeta = \frac{y}{b}, \quad \tau = \omega t \tag{41}$$

The equations of flight (8) take the form

$$\ddot{\xi} = -\frac{d^2\Phi_1(\tau)}{d\tau^2} - G \tag{42}$$

$$\ddot{\zeta} = -\frac{d^2\Phi_2(\tau)}{d\tau^2} - \Gamma \tag{43}$$

The conditions at the shock must be added to these equations:

$$\dot{\xi}_+ = (1-\lambda)\,\dot{\xi}_- \tag{44}$$

$$\dot{\zeta}_+ = -R\dot{\zeta}_- \tag{45}$$

If $\Phi_2(\tau) = \cos\tau$, then Eq. (43) coincides with Eq. (31) Sec. 41 in which the parameter p should be replaced by Γ. Equation (42) is linked with Eq. (43) only by the boundary conditions at the shock. Such a problem lends itself completely to solution at least in the case when there is one shock per period or per several periods of the exciting force. The corresponding solution of Eq. (43) takes the form

$$\zeta(\tau) = -\frac{\Gamma(\tau-\tau_0)^2}{2} - \Phi_2(\tau) + \Phi_2(\tau_0) + \pi n\Gamma(\tau-\tau_0) \tag{46}$$

the phase of the shock τ_0 being determined by the relation

$$\frac{d\Phi_2(\tau_0)}{d\tau} = \frac{1-R}{1+R}\pi n\Gamma \tag{47}$$

At $\Phi_2(\tau) = \cos\tau$ these two expressions coincide with expressions (36) and (34), Sec. 41, respectively. Assuming that the conditions for the existence and stability of the solution of Eq. (46) are satisfied (they can always be found with the aid of the procedure described in Sections 40 and 41), we can determine the value of the mean velocity of conveying.

The solution of Eq. (42) can be written in the following form:

$$\dot{\xi} = -\frac{d\Phi_1(\tau)}{d\tau} - G(\tau-\tau_0) + C_1 \tag{48}$$

where C_1 is a constant.

Introducing the solution into the condition (44) at the shock which, because of the periodicity of the solution, takes the form

$$\dot{\xi}(\tau_0) = (1-\lambda)\,\dot{\xi}(2\pi+\tau_0)$$

we obtain

$$C_1 = \frac{d\Phi_1(\tau_0)}{d\tau} - 2\pi nG\frac{1-\lambda}{\lambda} \tag{49}$$

Let us calculate now the mean value of the dimensionless velocity of conveying:

$$\dot{\xi}_{mean}=\frac{1}{2\pi n}\int_0^{2\pi n}\dot{\xi}\,d\tau=\frac{d\Phi_1(\tau_0)}{d\tau}-\pi nG\,\frac{2-\lambda}{\lambda} \tag{50}$$

Let the vibrating plane surface perform rectilinear oscillations, i.e., let $\Phi_1(\tau)=\Phi_2(\tau)$. Then, using expression (47), we find that

$$\dot{\xi}_{mean}=\frac{1-R}{1+R}\,\pi n\Gamma-\frac{2-\lambda}{\lambda}\,\pi nG \tag{51}$$

Thus the mean velocity of conveying for the regime of continuous tossing with rectilinear vibrations of the working member does not depend explicitly on the vibration law and is determined only by the parameter Γ, provided the conditions for the existence and stability of the solutions are satisfied. This reservation is essential as the distribution of the domains of the existence and stability of regimes with one shock per period along the Γ-axis in the parameter space is strongly dependent on the vibration law.

Note that with $R=0$ there are also domains of motions with continuous tossing. All the formulas obtained above for the case $R\neq 0$ hold only inside these domains.

Let us formulate the problem of the determination of the optimum vibration law for the working member whose motion is performed with continuous tossing up the particles. We restrict our treatment to the case of rectilinear vibrations of the horizontal plane surface, i.e., we set

$$\left.\begin{aligned} &x_0(t)=a\Phi_1(\omega t),\quad y_0(t)=b\Phi_1(\omega t)\\ &a=A_0\cos\beta,\quad b=A_0\sin\beta,\quad G=0\end{aligned}\right\} \tag{52}$$

In accordance with formula (51) and notations (18) the value of the mean velocity of conveying expressed in dimensional terms

$$V=a\omega\frac{1-R}{1+R}\,\pi n\Gamma=\frac{1-R}{1+R}\cdot\frac{\pi ng\cot\beta}{\omega} \tag{53}$$

Consequently, if Γ_1 and Γ_2 are two values of the parameter Γ, at which a regime is realized with the same n, and V_1 and V_2 are the corresponding values of the mean velocity, then with a constant A_0

$$\frac{V_1}{V_2}=\sqrt{\frac{\Gamma_1}{\Gamma_2}} \tag{54}$$

Therefore, in order to obtain an increase in the velocity of conveying it is necessary to seek such a vibration law with which the parameter Γ attains its highest value.

The same conclusion is reached if the problem is formulated as that of the maximum reducing of dynamic loads at the given mean

velocity of conveying. Let us assume, in distinction to (36), that the function $\Phi_1(\omega t)$ has been normalized according to the condition

$$\left|\frac{d^2\Phi_1(\tau)}{d\tau^2}\right|_{max} = 1 \tag{55}$$

In this case the maximum dynamic loads are proportional to $A_0\omega^2$.

Since at V = const ω must be constant, the decrease in loads will be achieved through the reduction of A_0, or as the parameter Γ is inversely proportional to A_0, through an increase in Γ, the regime of motion remaining the same.

Thus we have arrived at a peculiar variational problem: the functional that is to be maximized is not at all explicitly dependent on the vibration law.

Clearly the maximum value of the parameter Γ which corresponds to the regime with the given n is reached at the boundary of the domain of the existence and stability of this regime. Thus, for the sinusoidal law of vibration the optimum values of Γ are located at the right boundaries of the domains D_{1n} (Fig. 105).

Let us seek the optimum law among functions of the class which has the form

$$\Phi''(\tau) = \frac{1}{A_m}\left[\cos\tau + \rho\cos(3\tau + \varepsilon)\right] \tag{56}$$

where A_m = normalizing coefficient

ρ = ratio of the third harmonic amplitude to the amplitude of the first harmonic.

Such a vibration law can be realized either with the aid of a special vibration generator or by frequency multiplication using the method described in Section 34. The analytical determination of the boundaries for the domains of the existence and stability with arbitrary ρ and ε proves very difficult. As in the preceding cases, the required result can be quickly obtained by making use of an analog computer.

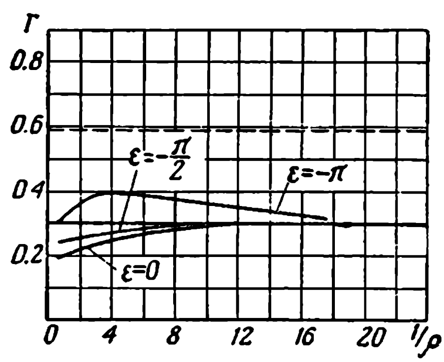

Figure 125

Figure 125 illustrates the relation between the maximum value of the parameter Γ_{max} corresponding to $n = 1$, $R = 0$ and ρ, ε; it has been obtained with the aid of a computer. The full horizontal line corresponds to the Γ_{max} value for purely sinusoidal vibrations, equal to 0.303. With increasing $1/\rho$ all the curves tend to this values irrespective of ε, as was to be expected.

The sharply defined maximum is attained with $\varepsilon = \pi/3$ and $1/\rho = 3$ to 4. In this case $\Gamma_{max} = 0.4$, i.e., the decrease in the dynamic loads that is obtainable at a constant velocity of conveying is 25% as compared with the case of purely sinusoidal vibrations.

Note that the expression

$$\cos\tau - \frac{1}{3}\cos 3\tau$$

corresponding to the values of ε and ρ determined above represents, with an accuracy of up to a constant factor, the two first harmonics of the expansion in a Fourier series of the stepwise-constant function

$$\Phi_1''(\tau) = \begin{cases} 1 \text{ at } 0 < \tau < \frac{\pi}{2} \text{ and } \frac{3}{2}\pi < \tau < 2\pi \\ -1 \text{ at } \frac{\pi}{2} < \tau < \frac{3}{2}\pi \end{cases} \tag{57}$$

The check carried out with the aid of an analog computer showed that with vibrations complying with the law (57) the loads can be halved as compared with those in the case of purely sinusoidal vibrations. The Γ_{max} value which corresponds to the law (57) is represented in Fig. 125 by the dash line.

Thus the stepwise-constant vibration law proves to be much more advantageous than its biharmonic approximation. A similar fact has been pointed out in discussing the optimum law of longitudinal vibrations.

To conclude, it should be pointed out that the results obtained by the theory of vibratory conveying are applicable at moderate velocities when the air resistance does not affect the motion considerably. Note also that the results that hold for the motion of individual particles must not be mechanically extended to cover the conveying of loose materials. The theory of this process, which is of great practical importance, is at present in the initial stage of development.

47. Effective Frequency of Vibration of Mixtures Containing Granular Aggregate

We shall use the following very simple model: a particle of granular aggregate is in direct contact with other solid particles and, in the general case, also with a liquid (in a concrete mixture, for example, the liquid is represented by the cement paste) and air bubbles. If the particle moves in relation to the surrounding medium mentioned, it is subjected, among other factors, to the action of applied forces of the dry friction type since the particle is in sliding contact with other solid particles. The more compact the mixture and the closer the particles of the aggregate, the larger the forces

of friction. Forces of the dry friction type can be partly generated by the plastic properties of a liquid, for example, of the cement paste. There may also arise the forces of viscous resistance, and elastic and adhesive forces.

If the mixture is acted upon by sinusoidal vibrations, its different particles will vibrate so that the amplitudes and phases of the fundamental tone, generally speaking, will be different and the frequency spectrum is enriched because of the nonlinearities. The motion of the given particle of the aggregate generated by the motion of the surrounding medium affects in its turn the motion of the medium in the neighbourhood of this particle. Consequently, in our very simple model the picture of the particle motion is rather complicated. In our approximate treatment the motion of all the elements of the surrounding medium will be assumed to be the same and independent of the motion of the particle being considered.

The compacting of mixtures with granular aggregate under the action of vibration is largely due to the pseudoliquid state mentioned in Section 44. When vibration provokes the slipping of some solid particles over others, then such a small constant force as weight proves able to displace the particles of the mixture wedged between the neighbouring ones and a closer packing of the particles results. This is realized by the apparent (sometimes by the actual) reduction of the coefficient of friction under the action of vibration.

In accordance with the results described in Section 44 the resistance offered by the medium to the motion of particle under the action of a small constant force will be the lower the wider the swing of the oscillation of the particle velocity in relation to the medium. Therefore we turn now to the determination of the relation between the half-swing of the velocity u_m of the particle in its relative vibratory motion and its dimensions, the vibration frequency ω and the velocity amplitude v_a of the absolute vibrations of the medium given by

$$v = v_a \cos(\omega t + \varphi) \tag{1}$$

Let us consider the one-dimensional problem. Assuming that the only kind of resistance to the motion of the particle is dry friction, we may write the differential equation of its relative vibration as follows:

$$\frac{du}{dt} = v_a \omega \sin(\omega t + \varphi) - \frac{F}{m} \operatorname{sgn} u \tag{2}$$

where F = constant modulus of friction force

m = mass of the particle

φ = initial phase of vibration of the medium at the moment $t = 0$ when the particle has the relative velocity $u = 0$ and the acceleration $du/dt > 0$.

We introduce the dimensionless parameters

$$\alpha = \frac{F}{mv_a\omega}; \quad \eta = \frac{u_m}{v_a} \tag{3}$$

It follows from what has just been stated that at a given amplitude of excitation velocity v_a the quantity η may be taken as a measure of the process effectiveness.

In the following discussion we shall consider only the vibrations of the particle at the excitation frequency. The analysis of Eq. (2) shows that continuous vibrations of this kind exist within the interval

$$0 \leqslant \alpha \leqslant \frac{2}{\sqrt{\pi^2+4}} \approx 0.573 \tag{4}$$

while within the interval

$$\frac{2}{\sqrt{\pi^2+4}} \leqslant \alpha < 1 \tag{5}$$

the particle makes, in its relative motion, two pauses of finite duration within the period $2\pi/\omega$ in the extreme positions. With $\alpha \geqslant 1$ there is no relative motion.

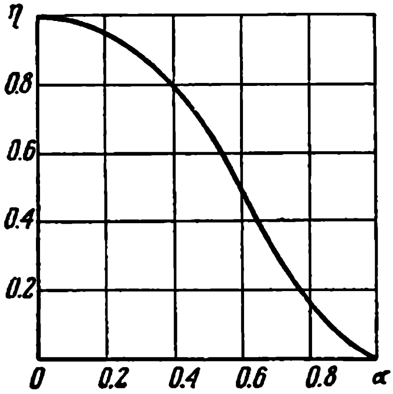

Figure 126

From the solution of Eq. (2) we conclude (cf. Section 20) that within the segment (4) the relation between η and α takes the form

$$\eta = \sqrt{1-\alpha^2} - \alpha\left(\frac{\pi}{2} - \cos^{-1}\alpha - \cos^{-1}\frac{\pi\alpha}{2}\right) \tag{6}$$

and within the interval (5)

$$\eta = 2\sqrt{1-\alpha^2} - \alpha(\pi - 2\sin^{-1}\alpha) \tag{7}$$

Figure 126 shows the graph of η vs. α plotted from expressions (6) and (7). It shows that the function $\eta(\alpha)$ decreases monotonically within the whole range $0 \leqslant \alpha < 1$.

If r is the characteristic linear dimension of the particle, its mass is then proportional to the cube of this dimension:

$$m = k_1 r^3 \tag{8}$$

where k_1 is a coefficient proportional to the particle density and taking account of its shape.

It is natural to take the force of resistance F to be proportional to the particle surface area since this area determines, on the average, the number of neighbouring particles with which the particle is in contact.

The surface area of the particle is proportional to the square of the characteristic dimension and so

$$F = k_2 r^2 \tag{9}$$

where k_2 is a coefficient depending on the shape and orientation of the particle.

Introducing (8) and (9) into the first of formulas (3), we obtain the relation between α and the characteristic dimension r:

$$\alpha = \frac{k}{v_a \omega r}, \quad \left(k = \frac{k_2}{k_1}\right) \tag{10}$$

If all the particles are considered to be geometrically similar, similarly oriented in space and to have the same density, then the coefficient k will be constant for particles of any size. In this case,

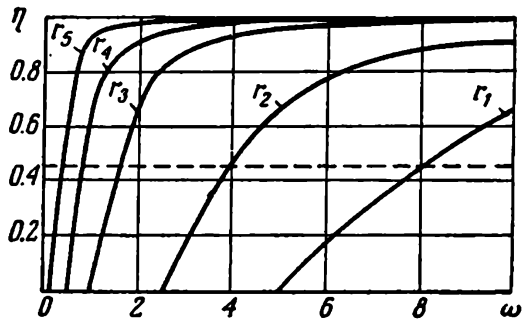

Figure 127

using relation (10) and taking into account that the function η (α) decreases monotonically, we conclude that the effectiveness of the process measured by η increases with increasing amplitudes of the excitation acceleration $v_a\omega$ and with increasing dimensions r of the particles. If the amplitude of the excitation velocity is constant, then in order to obtain the same effectiveness of vibration it is necessary to raise the vibration frequency for aggregate particles of smaller dimension.

For clarity the curves of η vs. the vibration frequency ω are shown in Fig. 127 at constant v_a for five values of r with $r_1 : r_2 : r_3 : r_4 : r_5 = 1 : 2 : 5 : 10 : 20$. The angular velocity ω is given in arbitrary units. Drawing a straight line parallel to the ω-axis at the required effectiveness level (the dash line in Fig. 127), we obtain the minimum frequencies necessary for sustaining the vibration of differently sized particles at the required intensity, the amplitude of the excitation velocity being the same. If a certain frequency ensures the necessary vibrations of particles of a given size, it surely will generate effective vibrations of particles having larger dimensions.

The above results will not be changed qualitatively if the viscosity of the medium and the conservative forces arising from the interaction between particle and medium will be taken into account.

The determination of the dependence of the effective vibration frequency on the size of the aggregate particles did not require the recourse to some hypothesis concerning different resonant frequencies for particles of different size. Such a hypothesis has been mentioned many times in publications abroad and in this country in connection with vibratory compacting of concrete mixtures, soils, powders, granular materials. This hypothesis cannot be accepted for the following reasons.

Firstly, in the media mentioned (possibly with the exception of fine powders) it is impossible to develop restoring forces that could give rise to natural frequencies of aggregate particles approaching the frequencies generated by the vibration machines. Neither the elastic forces resulting from the deformation of the solid and liquid ingredients of the medium, nor the forces of pressure of the air bubbles, nor the forces of surface tension of the liquid in the capillary interstices, nor the forces generated by the static electricity charges produced by vibration, can provide the natural frequencies mentioned above.

Secondly, the effect of the apparent reduction of the friction coefficients which enhances the compacting arises from the vibrational slipping of the particles accompanied by a considerable dissipation of energy. This excludes the possibility of a pronounced resonant peak appearing on the amplitude response curve of the relative vibrations of the particle with slipping, even in the presence of the required restoring force.

Thirdly, resonance is a selective phenomenon. If at a given frequency the effective vibration of particles of the same size is ensured by resonance, then larger or smaller particles in the mixture must vibrate with a much lower intensity, which would exclude the possibility of compacting the mixture. A simple harmonic vibration would be ineffective in such a case. However this is refuted by practice. On the other hand, vibration experiments with media containing grains of the same size have shown that there exists no unique frequency capable of providing a more effective compacting process in comparison with lower and higher frequencies.

CHAPTER 9

PROBLEMS OF VIBRATION ISOLATION

48. Vibration Isolation in Single-Degree-of-Freedom Systems

In discussing applied problems of vibration isolation they are often classified into two groups. Problems belonging to the first group refer to the protection of supporting or adjoining structures from the action of associated vibrating equipment containing a source of vibration. Problems of the second group refer to the protection of equipment from the effects of vibration of their supporting or adjoining structure. These problems are called by some authors the problems of "active isolation" and "passive isolation", respectively.

Such a division of the problems cannot be justified. In both cases we deal with the protection of passive objects that do not contain a source of vibration from the action of adjoining vibrating objects. Generally one supposes that in problems of the first group the vibrations are excited by a force and in those of the second, kinematically. Actually cases of kinematic as well as of force excitation are met with in problems of both groups.

Many practically important examples can be cited to show that the above-mentioned division of the problems of vibration isolation is not justifiable. We adduce two examples: the protection of a tractor hauling a vibrating roller from the latter's action and the protection of the operator's hand from the action of the vibrating control handle.

These statements do not preclude the expediency of using either the transmissibility of forces or that of displacements as criteria, the use of one or the other depending on the convenience of the treatment and the features of the concrete problem under consideration, though without any connection with the division of problems in the two groups mentioned. It will be shown later that the two criteria are mathematically identical.

It is natural to attempt first of all, if it is possible and permissible, to diminish the intensity of motion of a vibrating or striking object when the aim is the protection from vibrations or shocks. The intensity of vibration can be considerably reduced in some cases by adequate static and dynamic balancing of the mechanism; in

other cases *dynamic vibration absorbers* may prove effective; the theoretical basis of their operation has been discussed in Section 14.

When the further lowering of the intensity of motion of a vibrating object becomes unfeasible or inexpedient, vibration isolators must be selected to protect adjoining objects. Consider the case schematically pictured in Fig. 7. Let the sinusoidal exciting force (1), Sec. 7, be applied to body *1*. It is required that the vibration force transmitted to the fixed wall *3* be sufficiently small. Spring *2* is in this case an idealized vibration isolator. In accordance with formula (2), Sec. 6, the force Q transmitted by the isolator to the wall is defined by the expression

$$Q = cx \tag{1}$$

where c = coefficient of stiffness of vibration isolator

x = vibration displacement of body *1*.

We introduce now the transmissibility of the vibration force which is the ratio of the amplitude Q_a of the force Q to the amplitude F_a of the exciting force:

$$\eta = \frac{Q_a}{F_a} \tag{2}$$

From expression (6), Sec. 7, for the displacement amplitude we obtain

$$\eta = \frac{1}{\left|1 - \left(\frac{\omega}{\omega_0}\right)^2\right|} \tag{3}$$

where ω_0 is the natural frequency.

Using the dimensionless parameter γ defined by the second of expressions (3), Sec. 13, we obtain

$$\eta = \frac{1}{|1 - \gamma^2|} \tag{4}$$

The condition for the decrease of the force transmitted, i.e., for the inequality $\eta < 1$ being satisfied, is the inequality $\gamma > \sqrt{2}$. This condition remains also valid if there is any damping. The value of η is usually specified; then

$$\gamma = \sqrt{\frac{1}{\eta} + 1} \tag{5}$$

If isolator *3* possesses not only elastic but also damping properties, the system in Fig. 10 is obtained.

In this case the force Q is determined by the expression

$$Q = cx + b\dot{x} \tag{6}$$

where b is the coefficient of linear resistance. From (6) and taking into account expression (20), Sec. 7, for the vibration displace-

ment amplitude, we obtain

$$\eta=\sqrt{\frac{1+4\beta^2\gamma^2}{(1-\gamma^2)^2+4\beta^2\gamma^2}} \tag{7}$$

where β is defined by expression (43), Sec. 6.

Writing the force Q as a function of time, we have

$$Q=Q_a\cos(\omega t-\chi) \tag{8}$$

The initial phase

$$\chi=\tan^{-1}\frac{2\beta\gamma}{1-\gamma^2}-\tan^{-1}2\beta\gamma \tag{9}$$

Comparing formulas (7) and (9) with formulas (33), Sec. 13, we find that

$$\eta=\xi_{ka},\ \chi=\varphi_{da},\ (u=1,\ v=1) \tag{10}$$

i.e., that the force transmissibility and the initial phase of this force are equal to the dimensionless vibration amplitude for kinematic excitation and the initial phase of vibration, respectively, provided that $c_1=0$ and $b_1=0$[1] in the system in Fig. 19. This condition converts this system into system *13* in Fig. 35*b*. It follows that the force transmissibility η can be determined from Fig. 36*a* and the initial phase χ of the transmitted force, from Fig. 37*a*.

The problem of protecting an object from the effect of vibration of a supporting or adjoining structure performing a prescribed motion can also be treated within the bounds of system *13*, Fig. 35*b*. Consequently ξ_{ka} is the displacement transmissibility and φ_{da} the initial phase of the displacement of the object being isolated from vibration. Thus the two criteria of vibration isolation are defined by identical expressions.

At $\beta=0$ expression (7) takes the form (3) that is a special case of (7). Referring to expression (7) and to the family of graphs plotted from it in Fig. 36*a*, one can see that the vibration isolation becomes more effective with increasing γ and diminishing β. It follows that for better isolation of periodic vibration the isolators must ensure a sufficiently low natural frequency and a small damping. However, it is found practically reasonable to introduce a certain amount of damping to accelerate the decay of transient vibrations and prevent their excessive transient intensification, especially when the system passes through resonance or is subjected to strong jolts.

In the case of the isolation of low-frequency vibrations the isolators to be employed must have a very small stiffness, this requirement involving serious practical difficulties. This makes the system highly sensitive to the action of various static forces that cause considerable changes in the equilibrium position of the system when the isolators have a small stiffness. If the isolators serve at the

[1] Cf. equalities (31), Sec. 13.

same time as supports which carry the weight of the structure, then their static deformation determined by expression (16), Sec. 6, may prove too large to be tolerable. One way of attenuating such difficulties is the use of nonlinear vibration isolators having a softening restoring force characteristic.

The protection of objects from the action of single strong impulses, such as a fall from a great height or an explosion, has interesting features. In such cases one has usually to specify the value of permissible overload (acceleration measured in units of the acceleration of a freely falling body) which the object can withstand. The deformation of a protective device will be minimum when the resistance offered by it is constant and equal to the product of the mass of the object by the permissible acceleration.

A constant resistance is characteristic of plastic deformation and dry friction. A protective device based on the use of one of these factors will be subjected to permanent deformation. Vibration isolators with linear prestressed resilient elements can be used as protective devices capable of restoring its original state; this is important in the case of impulses applied many times. When the prestressing is high enough and the stiffness is small, the resistance offered by the isolator to deformation will be nearly constant. Another solution of the problem is the use of isolators with a softening restoring force characteristic.

49. Vibration Isolation in Multi-Degree-of-Freedom Systems

The static deformation of a linear resilient element or of a combination of such elements that has linear properties may be regarded only in two cases as the quotient obtained by dividing the applied static force by the stiffness of the resilient element (or the combination of elements), the static deformation being directed along the line of action of the force. The first case takes place when the displacement is possible only in one direction (systems having at least one degree of freedom). The second case is encountered when the force acts along one of the principal axes of stiffness of the resilient element (or the combination of elements) considered in Section 28.

In order to elucidate the deformation properties of the resilient element (isolator) when the direction of the force does not coincide with the principal axis of stiffness let us consider the plane arrangement in Fig. 128, where p and q are the principal axes of stiffness of resilient element *1* which is mounted on foundation *2*. The force F is applied to the free surface AB of the isolator along the y-axis inclined at the angle θ to the q-axis. On deformation of the isolator line AB remains straight and performs only translational motions.

Denoting the principal stiffnesses by c_p and c_q and projecting the force F on the p- and q-axes, we obtain the deformation of the

resilient element along these axes:

$$p=\frac{F\sin\theta}{c_p},\quad q=\frac{F\cos\theta}{c_q}$$

The determination of the deformation components in the directions of the x- and y-axes is now a simple matter:

$$y=p\sin\theta+q\cos\theta,\quad x=p\cos\theta-q\sin\theta$$

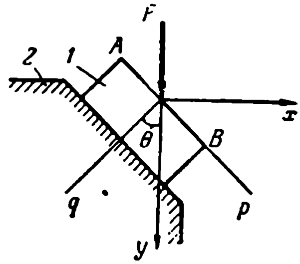

Figure 128

Hence

$$y=F\left(\frac{\sin^2\theta}{c_p}+\frac{\cos^2\theta}{c_q}\right),\quad x=\frac{1}{2}F\left(\frac{1}{c_p}-\frac{1}{c_q}\right)\sin 2\theta \tag{1}$$

From the first of equalities (1) we obtain

$$\frac{1}{c_y}=\frac{1}{2}\left(\frac{1}{c_q}+\frac{1}{c_p}\right)+\frac{1}{2}\left(\frac{1}{c_q}-\frac{1}{c_p}\right)\cos 2\theta \tag{2}$$

where c_y is the isolator stiffness in the direction of the line of action of the force.

In accordance with relation (2) the stiffness c_y reaches its extreme values c_q at $\theta=0$ and c_p at $\theta=\pi/2$; within the interval $0\leqslant\theta\leqslant\pi/2$ the stiffness is a monotonic function of θ. The second of equalities (1) shows that in the general case a displacement takes place along the x-axis, though there is no force acting in this direction. The deformation $x=0$ only at $\theta=0$ and $\theta=\pi/2$, which corresponds to the coincidence of the line of action of force F with one of the principal axes, or at $c_p=c_q$ when any direction may be taken to be the principal axis of stiffness.

The solution of the problem of vibration isolation in three dimensions even for the simplest case when the object to be protected can be idealized and regarded as a rigid body, i.e., when its complicated inner structure is disregarded, requires the consideration of six degrees of freedom. Generally speaking, this involves cumbersome calculations, beginning with the setting up of differential equations and the solution of a characteristic equation of degree twelve.

The problem would be greatly simplified if the free vibrations of each of the six coordinates of the system were not coupled with the vibrations of the other coordinates, i.e., if the normal coordinates were known. However, the determination of the normal coordinates may prove, as has been pointed out in Section 11, no less labour-consuming than the direct solution of the problem using the original system of coordinates. In the general case much work must also be done to find the direction of the principal axes of stiffness (and of the principal axes of damping in dissipative systems).

The problem is considerably easier if the system is symmetrical with respect to the coordinate axes and planes. In this case the differential equations fall into two uncoupled groups at least. The higher is the symmetry of the system, the larger the number of

independent groups of differential equations (the symmetry meant is that of the location of masses as well as of the elastic and dissipative elements).

The uncoupling, i.e., the elimination of the coupling of vibrations, is also of practical importance. It was shown in Section 48 that the vibration isolation becomes more effective with a reduced ratio of the natural frequency to the exciting frequency. It is practically difficult in many cases or even impossible to make all the six natural frequencies sufficiently low. Therefore, when measures are taken to provide a sufficiently small stiffness in the direction of one coordinate axis (or the angular stiffness relative to this axis) and the exciting force is directed along (or about) this axis, it is necessary to see that the vibrations of the coordinate concerned are not coupled with those of the other coordinates and do not draw into action other elastic forces that cannot be sufficiently decreased.

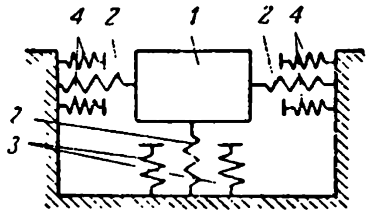

Figure 129

If the displacements of object *1* being isolated (see Fig. 129) are to be limited, relatively stiff elastic stops *3* on one side or stops *4* on two sides are installed in addition to the isolators *2* of large compliance. Clearances are left between the object being isolated and the stops. The relative displacement amplitudes under steady-state vibrations are ordinarily smaller than the clearances, and the stops remain inoperative. They come into play under transient regimes.

The use of elastic stops may lead to undesirable consequences. Thus, shocks against the stops occurring at starting may continue after the vibration excitation has become steady and they spell premature failure of the object being isolated. This reflects one of the features of nonlinear systems: the existence of a number of qualitatively different regimes and the dependence of the setting up of a certain regime on the initial conditions.

A steady-state shock-and-vibration regime may be harmonic (i.e. have the exciting frequency) or subharmonic. Such subharmonic regimes in single-degree-of-freedom systems have been discussed for the first time by Yorish. With stop-limiters on one side the subharmonic regimes may lead to a sharp increase in the vibration swing instead of keeping it within narrow limits.

In multi-degree-of-freedom systems steady shock-and-vibration regimes can set in with motion forms not observable in a system having no elastic stops (for instance, rotational vibrations may appear instead of translational ones).

These points should be taken into account at the stage of design of vibration isolation by investigating the behaviour of a nonlinear system with elastic limiters.

50. Sliding Behaviour and Simulation of Shock-and-Vibration Systems

The application of the point mapping method to the investigation of even the simplest shock-and-vibration systems calls for a considerable amount of calculations. This is seen from the examples presented in Sec. 41. The analytical study of a shock-and-vibration system with two degrees of freedom can no longer be brought up to finite formulas. Approximate analytical methods are most often found to be ineffective because of the loss of a multitude of qualitatively different periodic solutions characteristic of a shock-and-vibration system. There exists therefore a certain gap between the results of an analytical study of simplified models and practical problems. This refers, in particular, to the sliding regimes of movement of shock-and-vibration systems.

The gap in question can be closed up by constructing a mathematical model of the shock-and-vibration system, which is the algorithm of the solution of differential equations in the intervals between shocks and of the calculation of velocities at the moments of shocks. Such a model may be either a digital (realized on a digital computer) or an analog model.

The technique used for the analytical investigations differs, in principle, from simulation, independently of the type of model. The analytical investigation consists, in general, of two stages: (1) the construction of a solution of a certain definite type; (2) the determination of the domain of the existence and stability of this solution in the parameters space of the system by studying certain, often transcendental, expressions. The stages indicated are distinctly delineated. This allows the restrictions imposed on the movement of the system to be examined not all at a time but one after another, which simplifies the investigation to a certain extent.

In simulation, no question of the existence and stability of the system arises. The parameters of the system are assigned definite values and the integration is performed until a certain, previously unknown, periodic solution is obtained. The structure of the parameter space is elucidated by running over solutions for a large number of combinations of the parameters of the system. For this

purpose, it is necessary at the outset to include, in the model, all the possible limitations listed at the end of Sec. 40. An example is condition (3) in Sec. 41, which results in the appearance of the boundary C_τ.

Since such conditions are characteristic of all shock-and-vibration systems, we shall examine them in more detail. Let us consider a system in which the shock-and-vibration movement occurs in the coordinate ζ. Two kinds of motion can be realized in the system: either the motion of a body involving shocks against the fixed stop or the relative movement of two colliding bodies with a shock at $\zeta = \zeta_0$. At $\zeta = \zeta_0$ there takes place an instantaneous change in the structure of the system—the number of degrees of freedom decreases by unity. The subsequent behaviour of the system depends on the character of the forces acting on it. Let the velocity restitution coefficient, R, on shock be equal to zero. If, after the the shock $\ddot{\zeta} < 0$, then the value of the coordinate $\zeta = \zeta_0$ will be retained until the sign of $\ddot{\zeta}$ is changed. In a system with one degree of freedom, this situation corresponds to the domains D^0_{1n} of motions with pauses (see Fig. 103 in Sec. 41).

In the theory of automatic control the regimes under which the altered structure of the system is retained for a finite period of time are termed the sliding behaviour. This term is associated with the conceptions of the motion in the state space: the sliding behaviour is represented by the state trajectory, the portions of which lie entirely on definite state surfaces called discontinuity surfaces.

The state space of a single-mass non-autonomous shock-and-vibration system is presented in Fig. 99. On motion with pauses of finite duration ($R = 0$) the representative point, having arrived at the cylinder $\zeta = \zeta_0$ at its lower part corresponding to $\dot{\zeta} < 0$ turns instantly (i.e., along the generatrix) into a point on the circle $\dot{\zeta} = 0$ and "slides" along this circle up to the moment determined by expression (29) given in Sec. 41. Thus, the sliding behaviour occurs in the shock-and-vibration system.

At $R \neq 0$ the situation become somewhat complicated. Let us first consider a body that falls freely on the stop. If the absolute value of velocity just before the first collision is equal to v, then immediately after the subsequent collisions it will have the values of Rv, R^2v, etc. The time interval between the successive shocks will fall off in the same geometric progression, assuming the values of $2Rv/g$, $2R^2v/g$, etc., where g is the free-fall acceleration. From this it follows that the falling body will undergo an infinite number of collisions of decreasing force during a finite time period equal to $2Rv/(1 - R)g$.

Now let us return to the general case. If after the first shock $\ddot{\zeta} < 0$,

then at $R \neq 0$ there will follow an infinite sequence of shocks taking a certain finite time period, following which the system will pass over to the sliding behaviour. The possibility is not excluded, of course, that after a finite number of shocks of the indicated sequence the acceleration will become positive and the sliding behaviour will not set in.

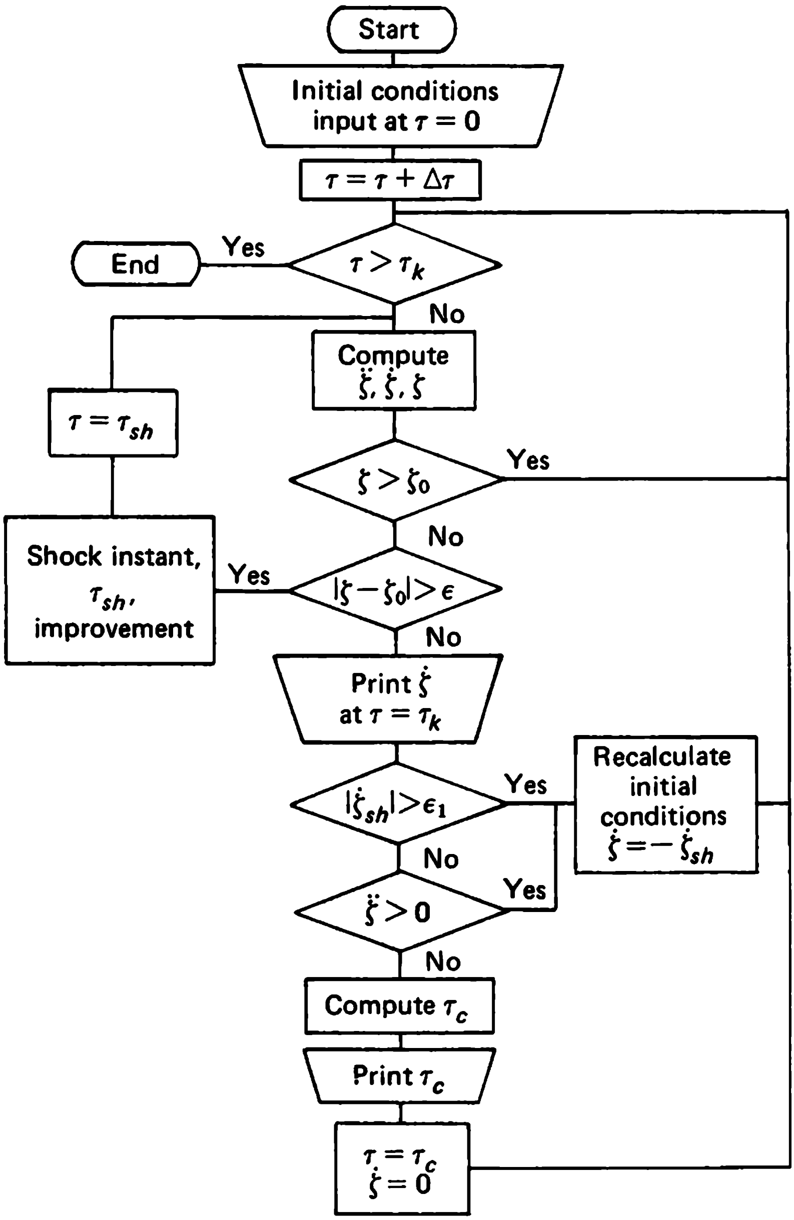

Figure 130

All these possible situations must be reflected in the model of the shock-and-vibration system, independently of the type of model. Figure 130 presents the block diagram of a digital model of the shock-and-vibration system of the general type. Integration of the equations of motion within the given interval $0 \leqslant \tau \leqslant \tau_k$ is carried out with a specified step $\Delta\tau$. After each integration step the condition $\zeta(\tau) > \zeta_0$ is checked up. If it has been fulfilled, the value of τ increases by an addend of $\Delta\tau$ and the integration is continued. If, however, $\zeta(\tau) < \zeta_0$ and $|\zeta(\tau) - \zeta_0|$ is greater than a certain specified small value of ϵ, the moment at which a shock occurs, τ_{sh}, is improved, i.e., an approximate solution of the equation $\zeta(\tau) - \zeta_0 = 0$ is sought for (e.g., by means of a linear interpolation). Then, the state coordinates of the system at $\tau = \tau_{sh}$ are calculated and, if necessary, recorded.

If at $\tau = \tau_{sh}$ the modulus of impact velocity exceeds a certain limiting value of ϵ_1, the initial conditions are recalculated and the integration of the equations of shockless motion is carried out. Otherwise, the sign of acceleration is checked up. At $\ddot{\zeta} > 0$, a transition to a new stage of shockless motion takes place. If $\ddot{\zeta} < 0$, it is assumed that the sliding stage has begun. The moment of completion of the sliding behaviour is then calculated, the state coordinates of the system are recorded and the next stage is started.

We shall not dwell here on the construction of analog models which must of necessity contain the logical elements that perform all the check-ups indicated.

The above-described algorithm allows one to solve the various problems that arise in the investigation of shock-and-vibration systems, including the study of the structure of the parameter space of the system, the examination of the steady-state motion dependence on the initial conditions, and the elucidation of the behaviour of the system with its parameters undergoing chance variations. To determine the periodic regimes, it is sufficient to incorporate an additional block which will perform the comparison of the state coordinates for successive shock moments.

51. Vibratory Conveying Without Gravitation

The vibratory conveying considered in Sec. 46 calls for the requisite participation of the gravity force to continuously press the moving body to the vibrating surface during the conveying without separation from the vibrating surface and to bring this body back to the vibrating surface during the conveying process involving tosses. In the absence of gravitation, this method of conveying is infeasible. Here we shall describe a different process of vibratory conveying, the

implementation of which requires neither gravitation nor any other external forces applied to the body being conveyed, except for the forces of its interaction with the vibrating surface.

Figure 131 shows a cylindrical body 1 not subjected to gravitation. It can move freely in a pipe 2 in the absence of any clearance between them. The Coulomb friction force may arise between the body and the pipe only when the transverse acceleration of the pipe develops.

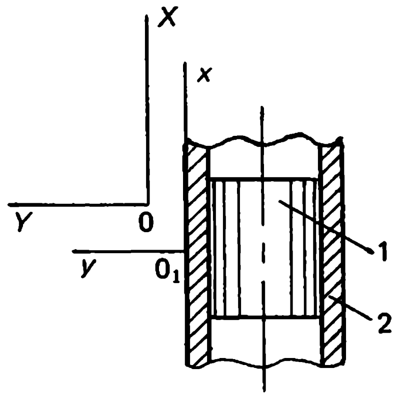

Figure 131

By analogy with Fig. 121 XOY is the fixed coordinate system; xO_1y is the coordinate system parallel to it, which is rigidly linked to the pipe. The coordinate systems indicated coincide when the pipe is placed in the middle position. Let the pipe perform a specified planar translational vibration according to the following law:

$$\begin{aligned} x_0(t) &= -a \cos 2\,\omega t, \\ y_0(t) &= b \cos(\omega t - \epsilon) \end{aligned} \tag{1}$$

that is, the frequency of longitudinal vibration is two times higher than the frequency of transverse vibration. The value of the initial phase, $-\epsilon$, is selected so that each point of the pipe describes a trajectory similar to the second Lissajou figure from the right in the second row of Fig. 5.

The transverse vibration of the pipe presses the body alternately to the right and left sides of the pipe surface. The simultaneously occurring longitudinal vibration causes the development of the variable friction force between the body and the pipe. With a certain set of ϵ values, the friction force produces a motion of the body

relative to the pipe, the mean velocity of which, $\dot{x}_m$, is greater than zero, i.e., is directed upwards.

The differential equation of the relative motion of the body may be written down in the following form:

$$m\ddot{x} = 4ma\omega^2 \cos 2\,\omega t + fmb\omega^2\ |\cos(\omega t - \epsilon)|\ \operatorname{sign}\dot{x} \qquad (2)$$

where m is the mass of the body, and f is the coefficient of dry friction.

There may be different motion regimes for the body being conveyed. Each cycle of the relative motion may consist of a series of upward and downward slipping regions and instantaneous or temporary pauses. The period of relative motion may be equal to or a multiple of the period of transverse vibration of the pipe.

Integration of the equation of relative motion can be carried out over domains, as was done in Sec. 46. The mean velocity of conveying, $\dot{x}_m$, must satisfy the condition

$$\int_0^{2n\pi} [4\cos 2\tau + \mu|\cos(\tau - \epsilon)|\operatorname{sign}\dot{x}]\,d\tau = 0 \qquad (3)$$
$$\operatorname{sign} 0 = 0, \quad (n = 1, 2, 3, \ldots)$$

where, by analogy with Eqs. (16) and (18) given in Sec. 46,

$$\tau = \omega t, \quad \mu = \frac{fb}{a} \qquad (4)$$

Equality (3) follows from the condition

$$\ddot{x}_m = 0 \qquad (5)$$

For the specified values of a and μ the mean velocity of conveying, $\dot{x}_m$, may reach a maximum value at a certain value of ϵ. A body of any shape, including a spherical one, can be conveyed by the indicated method in a pipe of any desired cross-sectional shape provided that the body touches the pipe without clearances at least at some of its points and cannot move in the transverse direction relative to the pipe. This method can be used to convey bulk materials, say, sand, gravel, peas, etc.

The conveying method described above is efficient not only without gravitation but also with counteracting gravitation, in which case the mean velocity of conveying will be lower. Thus, if the force of gravity $-mg$ is applied to body 1 in Fig. 131, the mean velocity of

vibratory conveying, $\dot{x}_m$, must satisfy the following condition

$$\int_0^{2n\tau} [4 \cos 2\tau + \mu | \cos(\tau - \epsilon) | \operatorname{sign} \dot{x} - G_0] \, d\tau = 0 \quad (6)$$

where

$$G_0 = \frac{g}{a \omega^2} \quad (7)$$

and g is the free-fall acceleration.

Such a conveying process under the influence of gravitation may not only be directed vertically upwards but also in a horizontal direction and upwards at any angle to the horizontal.

We have so far considered a motion without separation from the vibrating surface. This method can also be employed to convey body 1 with separation from the pipe walls 2 and alternate impacts against the opposite sides. This occurs if there is a clearance between the body and the pipe.

52. The Dynamics of Electromagnetic Vibration Exciters

The variable force that induces the vibrations of electromagnetic vibration exciters is the force of magnetic interaction of the periodically magnetized elements. The maximum magnitude of the exciting force of an electromagnetic vibration exciter is relatively low. Therefore, such exciters operate, as a rule, under near-resonance conditions, the amplitude of the force developed by the springs usually exceeding by 5-20 times the maximum value of the exciting force.

Two types of electromagnetic vibration exciters are differentiated: with movement of the armature relative to the core, which occurs along the principal direction of the magnetic flux and across the flux. On the other hand, a distinction is made between single-cycle electromagnetic vibration exciters, in which the electromagnetic exciting force acts only in one direction, and push-pull exciters in which the exciting force operates alternately in opposite directions.

A single-cycle electromagnetic vibration exciter with the longitudinal movement of armature 1 relative to the core 2 has one winding 3 (Fig. 132a) or two windings 3 and 3' (Fig. 132b). The armature and the core are interconnected by springs 4 and 4', which provide the near-resonance tuning of the system. The armatures and cores are made of a magnetically soft material. Alternating or pulsating voltage is applied to the ends of the windings and the current flowing in the windings induces the pulsating force of magnetic attraction

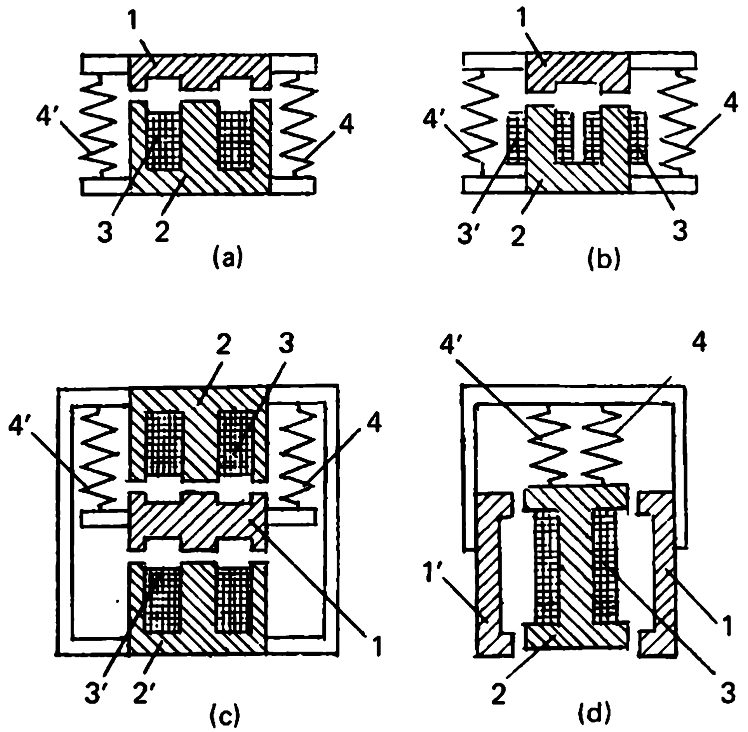

Figure 132

of the armature and core, which brings them closer together and causes the deformation of the springs. The reverse motion is effected at the expense of the potential energy stored in the springs.

A push-pull electromagnetic vibration exciter with a longitudinal motion of the armature 1 relative to the rigidly interlinked cores 2 and 2' have windings 3 and 3' and springs 4 and 4' (Fig. 132c). A voltage is applied alternately to the ends of the windings, which causes pulsating currents and appearance of magnetic attractive forces in the direct and the opposite course alternately.

Figure 132d shows a push-pull electromagnetic vibration exciter with the transverse motion of the rigidly interconnected armatures 1 and 1' relative to the core 2 bearing winding 3. The armatures are linked to the core via springs 4 and 4'. If a current pulse is induced in the winding in the position shown in Fig. 132d, the armatures begin moving under the action of the magnetic force directed upwards, thereby deforming the springs. Having passed the middle position, the armatures move under the inertia farther upwards more and more slowly and then begin moving down under the action of the deformed springs. Another current pulse is now generated in the winding and the armature is acted on by the magnetic force directed downwards, which accelerates their movement. This is followed by

the inertial retarded downward movement, the upward movement under the action of the deformed springs, etc.

Various current supply circuits are used for electromagnetic vibration exciters. We shall consider several simple current supply circuits for single-cycle vibration exciters. If an alternating voltage is applied to the ends of the windings (Fig. 133a), then two pulses of the magnetic attractive force will be generated per one voltage period in the magnetic system and, hence, the vibration frequency will be two times higher than the applied voltage frequency. The sign alternation of current pulses has no noticeable effect on vibration. The remagnetization of the core and armature nevertheless produces additional energy dissipation and warming; to avoid this, it is sometimes expedient to use a full-wave rectifier for current supply (Fig. 133b).

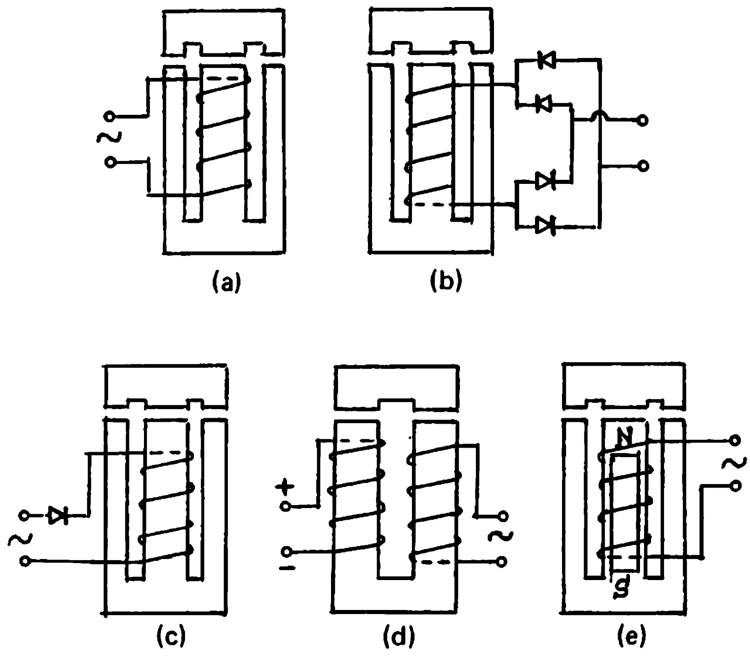

Figure 133

Figure 133c shows the current supply circuit from a half-wave rectifier. Here the frequency of forced vibration is equal to the applied voltage frequency. The same effect can be achieved by using an alternating-current mains if the core is fitted with an extra polarizing winding (Fig. 133d), to the ends of which there is applied a constant voltage, or if a permanent magnet (to be used for polarization) is incorporated into the core (Fig. 133e).

The simplied design scheme of a single-cycle electromagnetic vibration exciter is presented in Fig. 134. Here two inertia elements 1 and

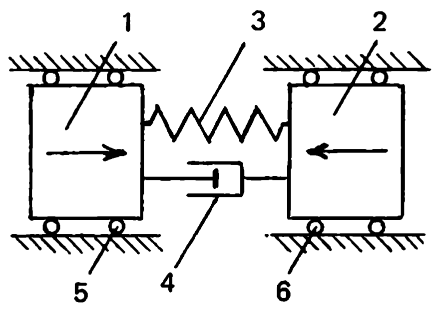

Figure 134

2, with masses m_1 and m_2, interconnected by means of a linear spring 3 and a damper 4, can move in ideal guides 5 and 6. The differential equations that describe the behaviour of this system may be written in the following form:

$$m_1 \frac{d^2 x_1}{dt^2} + b\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + c\,(x_1 - x_2) = F$$

$$m_2 \frac{d^2 x_2}{dt^2} - b\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) - c\,(x_1 - x_2) = -F \qquad (1)$$

$$\frac{d}{dt}(Li) + Ri = u$$

where t = time
b = resistance coefficient of damper
c = stiffness coefficient of spring
x_1, x_2 = coordinates of elements 1 and 2 measured from the equilibrium position
i = current in winding
L, R = inductance and electric resistance of winding
u = potential difference at the ends of windings
F = electromagnetic attractive force determined by the following relation:

$$F = \frac{\partial}{\partial x}\left(\frac{Li^2}{2}\right) \qquad (2)$$

where

$$x = x_1 - x_2 \tag{3}$$

Disregarding the magnetic reluctance of the steel magnetic circuit, which is considerably lower than the gap reluctances, we may write the following expression for the winding inductance:

$$L = L_0 \; [1 + (x/x_0)]^{-1} \tag{4}$$

where x_0 = width of each of gaps in the equilibrium position
L_0 = value of L at $x = 0$, which is given by the relation

$$L_0 = \mu_0 S_B w^2/x_0 \tag{5}$$

where $\mu_0 = 4\pi \cdot 10^{-7}\ H/m$ = magnetic permeability of vacuum, which is assumed to be equal to the magnetic permeability of air
w = number of winding turns,

$$S_B = \left[\sum_{k=1}^{n} (S_{Bk})^{-1} \right]^{-1} \tag{6}$$

S_{Bk} = cross-sectional area of the kth airgap
n = number of airgaps.

To take into account the magnetic leakage flux and the reluctance of the steel magnetic circuit, a correction factor α may be substituted in the right-hand side of equality (4):

$$L = L_0 \; \alpha^{-1} \; [1 + (x/x_0)]^{-1} \tag{7}$$

Hence, on the basis of expression (2), we obtain:

$$F = - \frac{L_0 i^2}{2\alpha x_0} \left[1 + \frac{x}{x_0} \right]^{-2} \tag{8}$$

Suppose that one mechanical degree of freedom determined by the coordinate x is realized in the system under consideration. Then, on the basis of Eqs. (1) with account taken of expressions (7) and (8),

we may write:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = -\frac{L_0 i^2}{2\alpha x_0}\left[1 + \frac{x}{x_0}\right]^{-2}$$

$$\frac{L_0}{\alpha}\frac{d}{dt}\left[i\left(1 + \frac{x}{x_0}\right)^{-1}\right] + Ri = u \tag{9}$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2} \tag{10}$$

If the current is supplied directly from an alternating-current mains (Fig. 133a),

$$u = u_a \sin \omega t \tag{11}$$

where u_a, ω = the amplitude and angular frequency of voltage.
To simplify the calculations, we pass over to dimensionless variables:

$$\tau = \omega t,\ \xi = \frac{x}{x_0},\ \eta = \frac{Ri}{u_a\,(1+\xi)} \tag{12}$$

and to the dimensionless parameters:

$$\rho = \frac{\alpha R}{L_0 \omega},\ \beta = \frac{b}{2m\omega}$$

$$\gamma = \frac{1}{\omega}\sqrt{\frac{c}{m}},\ \nu = \frac{L_0 u_a}{2\alpha x_0^2 m\omega^2 R^2} \tag{13}$$

and rewrite the differential equation (9), taking account of Eq. (11):

$$\ddot{\xi} + 2\beta\dot{\xi} + \gamma^2\xi = -\nu\eta^2$$

$$\dot{\eta} + \rho(1 + \xi)\eta = \rho \sin \tau \tag{14}$$

The dotes above the functions denote differentiation with respect to τ.

To integrate Eqs. (14), use is made of the method of successive approximations (see Sec. 17). At an initial approximation, we assume that

$$\xi = 0 \tag{15}$$

Then the integral of the second differential equation (14), which describes the steady-state periodic process, will have the following form:

$$\eta = \frac{\rho}{1+\rho^2} (\rho \sin\tau - \cos\tau) \tag{16}$$

or, with an accuracy of the values of third order of smallness in ρ,

$$\eta = -\rho(1 - \rho^2) \cos(\tau + \varphi_1) \tag{17}$$

where

$$\varphi_1 = \tan^{-1} \rho \tag{18}$$

Substituting expression (16) into the first of the differential equations (14), we obtain, after integration and elimination of the terms higher than the third order of smallness in ρ, the following relationship that describes the steady-state periodic vibration:

$$\xi = -\frac{\nu\rho^2(1-\rho)}{2\gamma^2} - \frac{\nu\rho^2}{2\sqrt{(\gamma^2 - 4)^2 + 16\beta^2}} \cos(2\tau + \varphi_2) \tag{19}$$

where

$$\varphi_2 = \tan^{-1} \frac{2\rho(\gamma^2 - 4) - 4\beta}{\gamma^2 - 4 + 8\rho\beta} \tag{20}$$

Let us consider the approximation obtained (19). The dimensionless quantity ξ, which is proportional to the relative displacement x, consists of a constant component, which is determined by the first term on the right-hand side of Eq. (19), and a sinusoidal component, which is determined by the second term and varies with frequency twice as great as the voltage frequency. Since the relative damping ratio is low ($\beta \ll 1$) and the system functions under the near-resonance regime ($\gamma \approx 2$), then $|\gamma - 2| \ll 1$ and $\sqrt{(\gamma^2 - 4)^2 + 16\beta^2} \ll 1$. Therefore the amplitude of the relative sinusoidal vibration is

considerably higher than the constant displacement. To avoid armature-core shocks, the following condition must be fulfilled:

$$\frac{\nu\rho^2}{2}\left(\frac{1-\rho}{\gamma^2}-\frac{1}{\sqrt{(\gamma^2-4)^2+16\beta^2}}\right)<1 \qquad (21)$$

In a special case, when $\gamma = 2$, expression (19) simplifies and assumes the form:

$$\xi = -\frac{\nu\rho^2(1-\rho)}{8}-\frac{\nu\rho^2}{8\beta}\cos\left(2\tau-\cos^{-1}2\rho\right) \qquad (22)$$

and condition (21) may be written thus:

$$\frac{\nu\rho^2}{8}\left(1-\rho+\frac{1}{\beta}\right)<1 \qquad (23)$$

According to the third of equations (12), the current in the winding is equal to

$$i=\frac{u_a}{R}\eta\,(1+\xi) \qquad (24)$$

Substituting the values of η and ξ from expressions (17) and (22) into Eq. (24) and neglecting the terms higher than the third order of smallness in ρ, we get:

$$i=\frac{\rho u_a}{R}\left\{\left[1-\frac{\nu\rho^2}{8}\left(1+\frac{1}{2\beta^2}\right)\right]\cos\left(\tau-\tan^{-1}\frac{1+2\rho^2}{\rho}\right)\right.$$
$$\left.-\frac{\nu\rho^2}{16\beta^2}\cos\left(3\tau-\tan^{-1}\frac{1-2\rho^2}{3\rho}\right)\right\} \qquad (25)$$

Hence, in the given approximation, the current contains the first and third harmonics.

The power required to sustain the vibration and to compensate for current losses is given by the following well-known relationship:

$$N=\frac{1}{2\pi}\int_0^{2\pi} ui\,d\tau \qquad (26)$$

Substituting the values of voltage and current from expressions (11) and (25) into relation (26), we have:

$$N = \frac{\rho u_a^2}{4\pi R} (1 - 0.5\rho^2) \left[1 - \frac{\nu\rho^2}{8}\left(1 + \frac{1}{2\beta^2}\right)\right] \quad (27)$$

If the current is supplied from a half-wave rectifier (Fig. 133c), then

$$u = \begin{cases} u_a \sin\tau & \text{at } 2n\pi \leqslant \tau \leqslant 2(n+1)\pi \\ 0 & \text{at } 2(n+1)\pi \leqslant \tau \leqslant 2(n+2)\pi \end{cases} \quad (28)$$
$$(n = 0, 1, 2, \ldots)$$

and, instead of the differential equations (14), we obtain:

$$\ddot{\xi} + 2\beta\dot{\xi} + \gamma^2\xi = -\nu\eta^2$$

$$\dot{\eta} + \rho(1+\xi)\eta = \begin{cases} \rho\sin\tau & \text{at } 2n\pi \leqslant \tau \leqslant 2(n+1)\pi \\ 0 & \text{at } 2(n+1)\pi \leqslant \tau \leqslant 2(n+2)\pi \end{cases} \quad (29)$$

Using the initial approximation (15), we obtain on the first interval $0 \leqslant \tau \leqslant \pi$ (assuming that $n = 0$):

$$\eta = \left(\eta_0 + \frac{\rho}{1+\rho^2}\right) e^{-\rho\tau} + \frac{\rho}{1+\rho^2}(\rho\sin\tau - \cos\tau) \quad (30)$$

where η_0 is the initial value of η at $\tau = 0$.

To determine η_0, we make use of the method of fitting (see Sec. 20). The finite value of η on the first interval at $\tau = \pi$:

$$\eta_1 = \left(\eta_0 + \frac{\rho}{1+\rho^2}\right) e^{-\pi\rho} + \frac{\rho}{1+\rho^2} \quad (31)$$

On the second interval $\pi \leqslant \tau \leqslant 2\pi$:

$$\eta = \eta_1 e^{-\rho(\tau-\pi)} \tag{32}$$

At the end of the second interval at $\tau = 2\pi$, from the condition of periodicity, η must be equal to η_0, i.e.,

$$\eta_0 = \eta_1 e^{-\pi\rho} \tag{33}$$

Substitution of the value of η_1 from expression (31) into Eq. (33) and solution of the resulting equation for η_0 gives

$$\eta_0 = \frac{\rho}{(1+\rho^2)(e^{\pi\rho}-1)} \tag{34}$$

By substituting the values of η into the first of the differential equations (29) and integrating it on the first and the second interval of τ, we can find $\xi(\tau)$ and, using the method of fitting, we can determine the initial values of ξ_0 and $\dot{\xi}_0$. Then, from expression (24) we can find the current $i(\tau)$. The power, N, may be calculated from the formula

$$N = \frac{1}{2\pi}\left[\int_0^{\pi} uid\tau + \int_{\pi}^{2\pi} uid\tau\right] \tag{35}$$

The procedure is rather simple but requires unwieldy computations.

If the current is supplied from a half-wave rectifier, the vibration frequency is equal to the frequency of the *a-c* mains, to which the rectifier is connected. The current in the winding is in this case richer in higher harmonics than in the case of current supply from the *a-c* mains. Nonetheless, the vibration is found to be practically sinusoidal due to the near-resonance tuning. Just as in the preceding case, the middle relative position of the vibrating parts is found to have been displaced from the equilibrium position by an amount equal to the ratio of the mean value of the magnetic attractive force to the stiffness coefficient of the springs.

An investigation of the dynamics of push-pull electromagnetic vibration exciters can be carried out in an analogous manner.

53. Fundamentals of Vibration Measurement by Means of Inertia Devices

Devices used for vibration measurements may be classified into two groups:

1. Devices which are used to measure vibrations relative to a reference point not connected with the vibrating object.
2. Inertia devices connected with the vibrating object and containing an elastically linked inertia element, the deformation of the elastic linkage is to be measured.

The scheme of an inertia device used for measuring vibrations is presented in Fig. 135. An inertia element 4 is linked to the vibrating

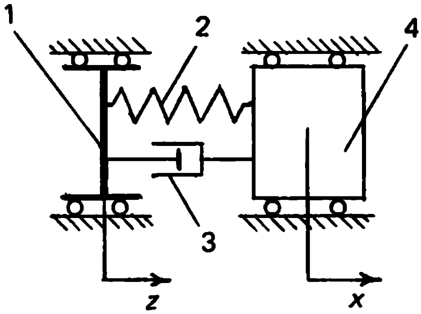

Figure 135

object 1 via spring 2 and damper 3. The scheme under consideration is identical with scheme 13 presented in Fig. 35 and is a special case of the circuit given in Fig. 19. If the vibrating object and the inertia element move along a straight line without turns, then, on the basis of Eq. (6) given in Sec. 8, we may write the following differential equation for the absolute motion of the inertia element:

$$m\ddot{x} + b\,(\dot{x} - \dot{z}) + c\,(x - z) = 0 \tag{1}$$

where m and x = mass and coordinate of the inertia element measured from the middle position

z = $z(t)$ = absolute coordinate of the vibrating object measured from the middle position, the movement of which is independent of the action of spring and damper

b = resistance coefficient of damper

c = stiffness coefficient of spring

t = time.

The dots above the functions signify differentiation with respect to time.

The deformation of the spring to be measured is equal to the relative coordinate:

$$y = x - z \tag{2}$$

Using expressions (1) and (2), we may write a differential equation for the relative movement of the inertia element:

$$m\ddot{y} + b\dot{y} + cy = -m\ddot{z} \tag{3}$$

We shall assume that the motion of the vibrating object is periodic. The periodic function $z(t)$ may be represented as the sum of a finite or infinite series of sinusoidal terms (see Sec. 4). Since Eqs (1) and (3) idealize the inertia device as a linear system, then, on the basis of the principle of superposition, the response of the device to the action expressed by the function $z(t)$ is equal to the sum of responses to the action of its sinusoidal terms (see Sec. 9).

If

$$z = z_a \cos \omega t \tag{4}$$

the differential equation (3) takes the form:

$$m\ddot{y} + b\dot{y} + cy = mz_a\omega^2 \cos \omega t \tag{5}$$

The integral of this equation, which describes the steady-state periodic forced vibration,

$$y = y_a \cos (\omega t - \varphi) \tag{6}$$

where, in accordance with Eqs. (24) and (21) given in Sec. 7 and also with Eq. (29) presented in Sec. 6,

$$y_a = \frac{z_a\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4h^2\omega^2}} \tag{7}$$

$$\varphi = \tan^{-1} \frac{2h\omega}{\omega_0^2 - \omega^2} \tag{8}$$

$$\omega_0 = \sqrt{c/m}, \quad h = b/2m \tag{9}$$

Using the dimensionless quantities determined by Eqs. (3) and (34) given in Sec. 13, i.e.,

$$\beta = h/\omega_0, \ \gamma = \omega/\omega_0, \ \tau = \omega_0 t, \ \eta = y/z_a \tag{10}$$

we obtain from relations (6), (7), and (8):

$$\eta = \eta_a \cos(\gamma\tau - \varphi) \tag{11}$$

$$\eta_a = \frac{\gamma^2}{\sqrt{(1-\gamma^2)^2 + 4\beta^2\gamma^2}} \tag{12}$$

$$\varphi = \tan^{-1} \frac{2\beta\gamma}{1-\gamma^2} \tag{13}$$

The amplitude-frequency characteristics of this system $\eta_a(\sigma)$, $\dot{\eta}_a(\sigma)$, $\ddot{\eta}_a(\sigma)$, are given in Figs. 40a, 42a, and 43a; the phase-frequency characteristic $\varphi(\sigma)$ is presented in Fig. 41a.

An inertia device can be used to directly measure the displacement as well as acceleration or velocity, depending on the relation between the frequency ω of the vibration measured, the natural frequency ω_0 of the undamped vibration of the inertia springed element and the damping coefficient h. To measure the vibration displacement by means of an inertia device, it is necessary that the following conditions be satisfied:

$$\omega_0 \ll \omega, \quad h \ll \omega \tag{14}$$

This can easily be checked up by writing Eqs. (7) and (8) in the following form:

$$y_a = \frac{z_a}{\sqrt{[(\omega_0/\omega)^2 - 1]^2 + (2h/\omega)^2}} \tag{15}$$

$$\varphi = \tan^{-1} \frac{2h/\omega}{(\omega_0/\omega)^2 - 1} \tag{16}$$

When conditions (14) are satisfied, we obtain:

$$z_a \approx y_a, \quad \varphi \approx \pi \tag{17}$$

Hence, the spring deformation, y, measured is nearly equal to the displacement, z, of the vibrating object.

To measure the vibration acceleration by means of an inertia device, it is necessary that the following conditions be fulfilled:

$$\omega_0 \gg \omega, \qquad h \ll \omega_0^2/\omega \tag{18}$$

In order to verify this, we write Eqs. (7) and (8) in the following form:

$$y_a = \frac{z_a \omega^2}{\omega_0^2 \sqrt{[1 - (\omega/\omega_0)^2]^2 + (2h\omega/\omega_0^2)^2}} \tag{19}$$

$$\varphi = \tan^{-1} \frac{2h\omega/\omega_0^2}{1 - (\omega/\omega_0)^2} \tag{20}$$

When conditions (18) are satisfied, we get:

$$\ddot{z}_a = z_a \omega^2 \approx \omega_0^2 y, \quad \varphi \approx 0 \tag{21}$$

Thus, the spring deformation, y, measured is nearly proportional to the acceleration, $\ddot{z}$, of the vibrating object.

In order to measure the vibration velocity by means of the inertia device, the following conditions must be fulfilled:

$$|\omega_0^2 - \omega^2| \ll 2h\omega \tag{22}$$

This follows from Eqs. (7) and (8) written down in the form:

$$y_a = \frac{z_a \omega}{2h\sqrt{[(\omega_0^2 - \omega^2)/2h\omega]^2 + 1}} \tag{23}$$

$$\varphi = \tan^{-1} \frac{2h\omega}{\omega_0^2 - \omega^2} \tag{24}$$

If the condition (22) is satisfied, we obtain:

$$\dot{z}_a = z_a\omega \approx 2hy_a, \qquad \varphi \approx \pi/2 \tag{25}$$

Hence, the spring deformation, y, measured is nearly proportional to the velocity, $\dot{z}$, of the vibrating object.

The dynamic errors arising in displacement, acceleration and velocity measurements decrease as the inequalities (14), (18), and (22), respectively, are increased.

The parameters of inertia devices usually satisfy either conditions (14) or (18). They very seldom satisfy condition (22) since in this case the frequency range of vibration measured is narrow or the sensitivity of the device is low.

CONCLUSION

54. Progress of Vibration Engineering and Problems of Research

The fields of application of vibration engineering are steadily increasing in number and more and more processes are employing vibration. The most important of them are the following:

(a) vibration moulding of reinforced-concrete components on platform vibrators, vibratory installations, vertical mould batteries;

(b) compaction of concrete mixtures by vibration in cast-in-situ structures by means of immersion and surface vibrators;

(c) compacting of soil and road beds by vibrotampers; tamping of asphalt-concrete road surfaces by vibratory rollers; compacting and smoothing of cement-concrete road surfaces by vibratory finishers;

(d) vibration and shock-and-vibration drilling of engineering-geological wells;

(e) handling of bulk materials by vibratory conveyers;

(f) feeding of space-oriented workpieces to automatic machine tools from vibrating hoppers;

(g) separation of materials according to size, density, shape, and friction coefficient by vibrating sieves, screens, separators, concentration tables;

(h) making of foundry moulds and cores by vibratory moulding presses;

(i) shaking-out of foundry flasks on vibrating grids;

(j) vibratory tumbling of forgings, stampings, castings;

(k) vibration polishing and precision finishing of machine and instrument parts;

(l) intensification of extraction processes in pulsating installations.

There has been a tendency of late to use large equipment, to increase its power and efficiency and improve quality indices. Vibration engineering and technological processes are relatively new fields with important problems still to be solved. Large-scale theoretical and experimental research is needed to solve these problems and to ensure further progress.

Some of the problems requiring further research are as follows:
(1) the generation of mechanical vibrations;
(2) the dynamics of vibration and shock-and-vibration machines;
(3) the properties of the various media acted upon by the working members of vibration machines;
(4) the interactions of working members and working media;
(5) the energy balance of a vibration machine;
(6) methods and systems for automation of the operation of vibration machines;
(7) determination of optimum operating conditions;
(8) reduction of the deleterious effects of vibrations on operating personnel and supporting structures.

The first group of problems associated with study of the processes of generating mechanical vibrations includes:
(a) investigation of vibration generators as devices transforming the energy of a source into mechanical vibrations;
(b) designing of generators of mechanical vibrations of prescribed form and spectrum, including random vibration generators;
(c) development of methods to multiply and demultiply vibration frequencies.

The problems involved in investigating the dynamics of vibration and shock-and-vibration machines include:
(a) study of transient processes and development of methods (i) to reduce starting power; (ii) to reduce the weight of drive elements; and (iii) to suppress excessive increase of vibration amplitude during passage through an intermediate resonance;
(b) study of the dynamics of resonant and near-resonant systems having one or more degrees of freedom;
(c) study of the dynamics of systems with subharmonic regimes, including shock-and-vibration systems;
(d) study of the dynamics of superharmonic systems;
(e) investigation of the dynamics of systems with distributed parameters and of combined (lumped-and-continuous) systems;
(f) solution of problems of the concurrent operation of two or more vibration generators.

Investigations of the properties of the various media acted upon by the working members of vibration machines comprises research into the following:
(a) the dynamics of granular and powdery media as regards problems of vibratory separation, conveying, mixing and compacting;
(b) soil dynamics as concerns problems of vibratory compacting, cutting, pile driving, and drilling;
(c) investigation of the dynamics of concrete mixtures as concerns problems of vibratory mixing, conveying, placing, compacting, moulding;

(d) plastic deformation of metals under the action of vibration and frequent blows in relation to vibration and shock-and-vibration pressing, stamping, drawing, rolling;

(e) the dynamics of asphaltic concretes as concerns problems of vibratory compaction;

(f) the mechanism of the influence of vibration on the physicochemical and physicomechanical processes in a liquid medium and at the interface between the liquid and solid phases, as concerns problems of the use of vibration for the processes of solution, extraction, emulsifying, sedimentation, filtration, leaching, etching, electrolysis, washingout, colouring, etc.

(g) the processes of vibratory cutting of metals, plastics, and other materials used in machine building as concerns problems of vibratory turning, milling, grinding, polishing, tumbling, etc.

The study of interactions between the working members of vibration and shock-and-vibration machines and the working medium, especially of force interactions concerns research into the following:

(a) the distribution of the forces and pressures applied to the working member by the medium; separate studies of normal and tangential acting factors;

(b) the physical nature of the acting forces, including dissipative, position, inertia, and combined forces;

(c) changes in the interaction forces under the influence of the vibrations of the working member and the medium.

The fifth group embraces problems involved in study of the energy balance of a vibration machine, of the circulation of energy flows, and of the character of energy dissipation in the system, and also of the relation between the behaviour of a vibration machine and the properties of the energy source.

The problems of automation include research and development work on methods and systems for automating vibration machines operation, self-tuning, programming, and the automation of quality control of the operation of vibration engineering equipment.

The seventh group includes the problems of determining the optimum operating conditions for vibration machines, investigation of their operational stability and the development of ways to improve the stability of their operation.

The eighth group includes problems of reducing the harmful effects of vibration and shock-and-vibration equipment on operating personnel and supporting structures.

This list, of course, does not touch on problems of design and purely empirical trends.

55. Classification of Vibration Machines

Principles of classification of vibration machines coinciding in their main outlines with those formulated below were put forward by the author in a paper to the Scientific and Technical Conference on Vibration Stands and Vibration Measurement in Leningrad in July 1959.

It is difficult to propose a universal clássification suitable for every case encountered and providing for all aspects of the design and application of vibration machines. Such a detailed classification would quickly become obsolete since vibration engineering is developing so rapidly. It is therefore not the classification itself but its basic principles and trends that are of general interest. Guided by these principles one can draw up detailed classifications for particular groups of vibration machines.

The principles for classifying vibration machines that we suggest are as follows.

1. By *purpose*. A classification according to purpose can be made for large groups of general-purpose machines or for special-purpose machines. Vibration machines can broadly be divided, for example, into machines for compacting, loosening, mixing, separating, conveying, etc. Special-purpose machines may be divided into special types, such as platform vibrators for moulding reinforced-concrete components, vibrating hoppers for feeding parts to automatic machine tools, general-purpose vibration generators which are fixed to the object, internal vibrators for compacting concrete mixtures, vibratory grids for shaking out foundry flasks.

2. By *type of drive*. Machines with electric, hydraulic, pneumatic drives, and machines driven by internal combustion engines are distinguished. A further subdivision can be made by type of drive.

3. By *type of transformation of the supply energy into mechanical vibrations*: centrifugal, piston, eccentric, crank-gear, electromagnetic, electrodynamic, magnetostriction, piezoelectric, pulsating, self-induced, kinematically-excited vibration machines, etc. Thus, this subdivision involves multiplicity of types in each group.

4. By *number of vibrating rigid bodies*: single-mass, two-mass, three-mass, etc., machines.

5. By *shape of vibration of the working member*. Machines with rectilinear vibrations, elliptic vibrations, helical vibrations, various combined vibrations, etc.

6. By *periodicity of vibration*: simple periodic, modulated, almost periodic, non-periodic (random) vibration machines.

7. By *the spectrum of periodic vibration of the working member*: sinusoidal, biharmonic, polyharmonic vibration machines.

8. By *presence of shock*: shockless, shock-and-vibration machines with shocks of the first order, second order, third order.

9. By *the relation between the exciting and natural frequencies*: preresonance, postresonance, near-resonance, interresonance vibration machines.

10. By *the number of vibration generators on the working member*: machines with one generator, two generators, etc.

11. By *method of synchronizing the operation of vibration generators*: positive mechanical synchronization, positive electrical synchronization, self-synchronization, without synchronization.

12. By *frequency range*: high-frequency, medium-frequency, low-frequency machines. The designations are relative and depend on the type of technological process and the type of vibration machine.

13. By *method of control*: no control, manual adjustment, mechanical regulation, automatic control, programmed control, self-tuning for the optimum operating conditions.

14. By *degree of fixation of the kinematic parameters of the working member*: completely positive motion of the working member (positive kinematics), partly positive motion of the working member, for example, along a definite direction (semi-positive kinematics), no rigid restraints of the working member (dynamic-type).

This list is far from exhaustive as regards possible trends of classification and scarcely touches on certain design and service features of vibration machines. But many of the features listed are essential for most vibration machines and are given here in order to unify the classification principles for specialized machine groups.

CONCLUSION

54. Progress of Vibration Engineering and Problems of Research

The fields of application of vibration engineering are steadily increasing in number and more and more processes are employing vibration. The most important of them are the following:

(a) vibration moulding of reinforced-concrete components on platform vibrators, vibratory installations, vertical mould batteries;

(b) compaction of concrete mixtures by vibration in cast-in-situ structures by means of immersion and surface vibrators;

(c) compacting of soil and road beds by vibrotampers; tamping of asphalt-concrete road surfaces by vibratory rollers; compacting and smoothing of cement-concrete road surfaces by vibratory finishers;

(d) vibration and shock-and-vibration drilling of engineering-geological wells;

(e) handling of bulk materials by vibratory conveyers;

(f) feeding of space-oriented workpieces to automatic machine tools from vibrating hoppers;

(g) separation of materials according to size, density, shape, and friction coefficient by vibrating sieves, screens, separators, concentration tables;

(h) making of foundry moulds and cores by vibratory moulding presses;

(i) shaking-out of foundry flasks on vibrating grids;

(j) vibratory tumbling of forgings, stampings, castings;

(k) vibration polishing and precision finishing of machine and instrument parts;

(l) intensification of extraction processes in pulsating installations.

There has been a tendency of late to use large equipment, to increase its power and efficiency and improve quality indices. Vibration engineering and technological processes are relatively new fields with important problems still to be solved. Large-scale theoretical and experimental research is needed to solve these problems and to ensure further progress.

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Subject Index

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